Pricing without martingale measure

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Abstract: For several decades, the no-arbitrage (NA) condition and the martingale measures have played a major role in the financial asset’s pricing theory. Here, we propose a new approach based on convex duality instead of martingale measures duality: our prices will be expressed using Fenchel conjugate and bi-conjugate. This naturally leads to a weak condition of (NA) called Absence of Immediate Profit (AIP). It asserts that the price of the zero claim should be zero or equivalently that the super-hedging cost of some call option should be non-negative. We propose several characterizations of the (AIP) condition and also study the relation with (NA) and a stronger notion of (AIP) linked to the no-free lunch condition. We show in a one step model that under (AIP) the super-hedging cost is just the payoff’s concave envelop. In the multiple-period case, for a particular, but still general setup, we propose a recursive scheme for the computation of a the super-hedging cost of a convex option. We also give some promising numerical illustrations.

Keywords and phrases: Financial market models, Super-hedging prices, No-arbitrage condition, Conditional support, Essential supremum.

1 Introduction

The problem of giving a fair price to a financial asset $G$ is central in the economic and financial theory. A selling price should be an amount which is enough to initiate a hedging strategy for $G$, i.e. a strategy whose value at maturity is always above $G$. It seems also natural to ask for the infimum of such amount. This is the so called super-replication price and it has been introduced in the binomial setup for transaction costs by [7]. Characterising and computing the super-replication price has become one of the central issue in
mathematical finance theory. Until now it was intimately related to the No-Arbitrage (NA) condition. This condition asserts that starting from a zero wealth it is not possible to reach a positive one (non negative almost surely and strictly positive with strictly positive probability measure). Characterizing the (NA) condition or, more generally, the No Free Lunch condition leads to the Fundamental Theorem of Asset Pricing (FTAP in short). This theorem proves the equivalence between those absence of arbitrage conditions and the existence of equivalent risk-neutral probability measures (also called martingale measures or pricing measures) which are equivalent probability measures under which the (discounted) asset price process is a martingale. This was initially formalised in [15], [16] and [21] while in [12] the FTAP is formulated in a general discrete-time setting under the (NA) condition. The literature on the subject is huge and we refer to [13] and [19] for a general overview. Under the (NA) condition, the super-replication price of $G$ is equal to the supremum of the (discounted) expectation of $G$ computed under the risk-neutral probability measures. This is the so called dual formulation of the super-replication price or Superhedging Theorem. We refer to [?] and [14] and the references therein.

Another approach to solve the super-hedging problem is to use so-called minimax strategies. The idea it to maximise the hedging error in the worst cases, i.e. we fix a finite number of possible values for the prices and we solve an optimization problem the solution of which is the strategy [4, 17]. Naturally, duality is still present in this formulation because of the dual problem. In finance, this is clearly a major topic, which is in particular discussed in [11]. This is also confirmed in our approach.

In this paper, a super-hedging or super-replicating price is the initial value of some super-hedging strategy. We propose an innovating approach: we analyse from scratch the set of super-hedging prices and its infimum value, which will be called the infimum super-hedging cost. Note that this cost is not automatically a super-replicating price. Under mild assumptions, we show that the one-step set of super-hedging prices can be expressed using Fenchel-Legendre conjugate and the infimum super-replication cost is obtained by the Fenchel-Legendre biconjugate. So, we use here the convex duality instead of the usual financial duality based on martingale measures under the (NA) condition. We then introduce the condition of Absence of Immediate Profit (AIP). An Immediate Profit is the possibility to super-hedge 0 at a negative cost. We prove that (AIP) is equivalent to the fact that the stock value at the beginning of the period belongs to the convex envelop of the conditional
support of the stock value at the end of the period. Using the notion of conditional essential supremum, it is equivalent to say that the initial stock price is between the conditional essential infimum and supremum of the stock value at the end of the period. Under (AIP) condition, we show that the one-step infimum super-hedging cost is the concave envelop of the payoff relatively to the convex envelop of the conditional support. We also show that (AIP) is equivalent to the non-negativity of the super-hedging price of any fixed call option. We then study the multiple-period framework. We show that the global (AIP) condition and the local ones are equivalent. We then focus on a particular, but still general setup, where we propose a recursive scheme for the computation of the super-hedging prices of a convex option. We obtain the same computative scheme as in [8] and [9] but here it is obtained by only assuming (AIP) instead of the stronger (NA) condition. We also give some numerical illustrations; we calibrate historical data of the french index CAC 40 to our model and implement the super-hedging strategy for a call option. Our procedure is in a sens model free and based only on statistical estimations.

Finally, we study the link between (AIP), (NA) and the absence of weak immediate profit (AWIP) conditions. We show that the (AIP) condition is the weakest-one and we also provide conditions for the equivalence between the (AIP) and the (AWIP) conditions, as well as characterization through absolutely continuous martingale measure.

The paper is organized as follows. In Section 2, we study the one-period framework while in Section 3 we study the multi-period one. Section 4 is devoted to the comparison between (AIP), (NA) and (AWIP) conditions. Section 5 proposes some explicit pricing for a convex payoff and numerical experiments. Finally, Section 6 collects the results on conditional support and conditional essential supremum.

In the remaining of this introduction we introduce our framework and recall some results that will be used without further references in the sequel. Let \( (\Omega, (\mathcal{F}_t)_{t \in \{0,\ldots,T\}}, \mathcal{F}_T, P) \) be a complete filtered probability space, where \( T \) is the time horizon. We consider a \( (\mathcal{F}_t)_{t \in \{0,\ldots,T\}} \)-adapted, real-valued, non-negative process \( S := \{S_t, t \in \{0,\ldots,T\}\} \), where for \( t \in \{0,\ldots,T\} \), \( S_t \) represents the price of some risky asset in the financial market in consideration. Trading strategies are given by \( (\mathcal{F}_t)_{t \in \{0,\ldots,T\}} \)-adapted processes \( \theta := \{\theta_t, t \in \{0,\ldots,T - 1\}\} \) where for all \( t \in \{0,\ldots,T - 1\} \), \( \theta_t \) represents the investor’s holding in the risky asset between time \( t \) and time \( t + 1 \). We assume that trading is self-financing and that the riskless asset’s price is a
constant equal to 1. The value at time \( t \) of a portfolio \( \theta \) starting from initial capital \( x \in \mathbb{R} \) is then given by

\[
V_t^{x,\theta} = x + \sum_{u=1}^{t} \theta_{u-1} \Delta S_u.
\]

For any \( \sigma \)-algebra \( \mathcal{H} \) and any \( k \geq 1 \), we denote by \( L^0(\mathbb{R}^k, \mathcal{H}) \) the set of \( \mathcal{H} \)-measurable and \( \mathbb{R}^k \)-valued random variables. Let \( h : \Omega \times \mathbb{R}^k \to \mathbb{R} \). The effective domain of \( h(\omega, \cdot) \) is \( \text{dom} h(\omega, \cdot) = \{ x \in \mathbb{R}^k, h(\omega, x) < \infty \} \) and \( h(\omega, \cdot) \) is proper if \( \text{dom} h(\omega, \cdot) \neq \emptyset \) and \( h(\omega, x) > -\infty \) for all \( x \in \mathbb{R}^k \). Next if \( h \) is \( \mathcal{H} \)-normal integrand (see Definition 14.27 in [26]) then \( h \) is \( \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k) \)-measurable and is lower semi-continuous (l.s.c. in the sequel, see [26, Definition 1.5]) in \( x \) and the converse holds true if \( \mathcal{H} \) is complete for some measure, see [26, Corollary 14.34]. Let \( Z \in L^0(\mathbb{R}^k, \mathcal{H}) \), we will use the notation \( h(Z) : \omega \to h(Z(\omega)) = h(\omega, Z(\omega)) \) and, if \( h \) is \( \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k) \)-measurable, then \( h(Z) \in L^0(\mathbb{R}^k, \mathcal{H}) \).

A random set \( \mathcal{K} \) is a \( \mathcal{H} \)-measurable if for all open set \( O \) of \( \mathbb{R}^k \), \( \{ \omega \in \Omega, O \cap \mathcal{K}(\omega) \neq \emptyset \} \in \mathcal{H} \). If \( \mathcal{K} \) is a \( \mathcal{H} \)-measurable and closed-valued random set of \( \mathbb{R}^k \), then \( \mathcal{K} \) admits a Castaing representation \((\eta_n)_{n \in \mathbb{N}}\) (see Theorem 14.5 in [26]) : \( \mathcal{K}(\omega) = \text{cl}\{ \eta_n(\omega), n \in \mathbb{N} \} \) for all \( \omega \in \text{dom} \mathcal{K} = \{ \omega \in \Omega, \mathcal{K}(\omega) \cap \mathbb{R}^k \neq \emptyset \} \), where the closure is taken in \( \mathbb{R}^k \).

2. The one-period framework

For ease of notation, we consider two complete sub-\( \sigma \)-algebras of \( \mathcal{F}_T \) : \( \mathcal{H} \subseteq \mathcal{F} \) and two random variables \( y \in L^0(\mathbb{R}, \mathcal{H}) \) and \( Y \in L^0(\mathbb{R}, \mathcal{F}) \). We assume that \( y(\omega) \) and \( Y(\omega) \) are non-negative for all \( \omega \in \Omega \). As \( P(Y \in \text{supp}_\mathcal{H} Y) = 1 \) (see Remark 6.2), we deduce that \( \text{supp}_\mathcal{H} Y \) is a.s. non-empty. For ease of notation, we will assume that \( \text{supp}_\mathcal{H} Y(\omega) \) is non-empty for all \( \omega \in \Omega \). Moreover, as \( 0 \leq Y < \infty \), \( \text{Dom supp}_\mathcal{H} Y = \Omega \). The setting will be applied in Section 3 with the choices \( \mathcal{H} = \mathcal{F}_t, \mathcal{F} = \mathcal{F}_{t+1}, Y = S_{t+1}, y = S_t \).

Section’s objective is to obtain a characterisation of the one-step set of super-hedging (or super-replicating) prices of \( g(Y) \) under suitable assumptions on \( g : \Omega \times \mathbb{R} \to \mathbb{R} \).

In the following, the notion of conditional support (\( \text{supp}_\mathcal{H} Y \)), conditional essential infimum (\( \text{ess inf}_\mathcal{H} \)) or supremum (\( \text{ess sup}_\mathcal{H} \)) will be in force, see Section 6.
Definition 2.1. The set $\mathcal{P}(g)$ of super-hedging prices of the contingent claim $g(Y)$ consists in the initial values of super-hedging strategies $\theta$:

$$\mathcal{P}(g) = \{ x \in L^0(\mathbb{R}, \mathcal{H}), \exists \theta \in L^0(\mathbb{R}, \mathcal{H}), x + \theta(Y - y) \geq g(Y) \text{ a.s.} \}.$$ 

The infimum super-hedging cost of $g(Y)$ is defined as $p(g) := \text{ess inf}_H \mathcal{P}(g)$.

Notice that the infimum super-hedging cost is not a priori a price, i.e. an element of $\mathcal{P}(g)$, as the later may be an open interval.

Lemma 2.2.

$$\mathcal{P}(g) = \\left\{ \text{ess sup}_H (g(Y) - \theta Y) + \theta y, \theta \in L^0(\mathbb{R}, \mathcal{H}) \right\} + L^0(\mathbb{R}_+, \mathcal{H}). \quad (2.1)$$

Suppose that $g$ is a $\mathcal{H}$-normal integrand. Then

$$\text{ess sup}_H (g(Y) - \theta Y) = \sup_{z \in \text{supp}_H Y} (g(z) - \theta z) = f^*(-\theta) \text{ a.s.} \quad (2.2)$$

where $f^*$ is the Fenchel-Legendre conjugate of $f$ i.e.

$$f^*(\omega, x) = \sup_{z \in \mathbb{R}} (xz - f(\omega, z)), \quad f(\omega, z) = -g(\omega, z) + \delta_{\text{supp}_H Y}(\omega, z), \quad (2.3)$$

where $\delta_C(\omega, z) = 0$ if $z \in C(\omega)$ and $+\infty$ else. Both $f^*(\omega, \cdot)$ and $x \to f^*(\omega, -x)$ are proper, convex, l.s.c., $f^*$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable and $f^*$ is a $\mathcal{H}$-normal integrand. Moreover, we have that

$$p(g) = -f^{**}(y) \text{ a.s.}$$

where $f^{**}$ is the Fenchel-Legendre biconjugate of $f$ i.e.

$$f^{**}(\omega, x) = \sup_{z \in \mathbb{R}} (xz - f^*(\omega, z)) .$$

Proof. As $x \in \mathcal{P}(g)$ if and only if there exists $\theta \in L^0(\mathbb{R}, \mathcal{H})$ such that $x - \theta y \geq g(Y) - \theta Y$ a.s., we get by definition of the conditional essential supremum (see Definition 6.4) that (2.1) holds true. Then (2.2) follows from Lemma 6.9. Lemma 6.3 will be in force. Since the graph of the closed-valued random set $\text{supp}_H Y$ belongs to $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$, we easily deduce that $\delta_{\text{supp}_H Y}$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable and l.s.c. As $\text{dom} f = \text{supp}_H Y$ is non-empty $f^*(\omega, \cdot)$ is convex and l.s.c. as the supremum of affine functions.
Hence \( x \rightarrow f^*(\omega, -x) \) is also l.s.c. and convex. Moreover, using Lemma 6.7, \( f^*(\omega, x) = \sup_{z \in \text{supp}_H Y(\omega)} (xz + g(\omega, z)) \) is \( \mathcal{H} \otimes \mathcal{B}(\mathbb{R}) \)-measurable.

\[
p(g) = \text{ess inf}_H \{ f^*(-\theta) + \theta y, \ \theta \in L^0(\mathbb{R}, \mathcal{H}) \} \quad \text{a.s.}
\]

\[
= -\text{ess sup}_H \{ \theta y - f^*(\theta), \ \theta \in L^0(\mathbb{R}, \mathcal{H}) \} \quad \text{a.s.}
\]

\[
= -\sup_{z \in \mathbb{R}} (zy - f^*(z)) = -f^{**}(y) \quad \text{a.s.}
\]

The first equality is a direct consequence of (2.1), the second one is trivial. In order to obtain the third one, we apply Lemma 6.10. Indeed remark first that \( \text{ess sup}_H \{ \theta y - f^*(\theta), \ \theta \in L^0(\mathbb{R}, \mathcal{H}) \} = \text{ess sup}_H \{ \theta y - f^*(\theta), \ \theta \in L^0(\mathbb{R}, \mathcal{H}) \cap \text{Dom } f^* \} \). Now since \( f^* \) is \( \mathcal{H} \otimes \mathcal{B}(\mathbb{R}) \)-measurable, we deduce that \( \text{graph } \text{dom } f^* = \{ (\omega, x) \in \Omega \times \mathbb{R}, \ f^*(\omega, x) < \infty \} \) is a \( \mathcal{H} \otimes \mathcal{B}(\mathbb{R}) \)-measurable set and \( \text{dom } f^* \) is also \( \mathcal{H} \)-measurable (see [26, Theorem 14.8]). Since \( (\omega, z) \rightarrow zy(\omega) - f^*(\omega, z) \) is a \( \mathcal{H} \otimes \mathcal{B}(\mathbb{R}) \)-measurable function and \( f^*(\omega, \cdot) \) is convex and thus u.s.c. on \( \text{dom } f^*(\omega) \), we may apply Lemma 6.10 and we obtain that a.s.

\[
\text{ess sup}_H \{ \theta y - f^*(\theta), \ \theta \in L^0(\mathbb{R}, \mathcal{H}) \cap \text{Dom } f^* \} = \sup_{z \in \text{Dom}(f^*)} (zy - f^*(z)) \quad \text{a.s.}
\]

\[
= \sup_{z \in \mathbb{R}} (zy - f^*(z)).
\]

\( \square \)

Let \( \text{conv } h \) be the convex envelop of \( h \) which is the greatest convex function dominated by \( h \), i.e. \( \text{conv } h(x) = \sup \{ u(x), \ u \text{ convex and } u \leq h \} \). The concave envelop is defined symmetrically and denoted by \( \text{conc } h \). We also define the (lower) closure \( \overline{h} \) of \( h \) as the greatest l.s.c. function which is dominated by \( h \) i.e. \( \overline{h}(x) = \lim_{y \to x} h(y) \). The upper closure is defined symmetrically: \( \overline{h}(x) = \lim_{y \to x} h(y) \). It is easy to see that

\[
\text{conv } f(y) = \sup \{ \alpha y + \beta, \ \alpha, \beta \in \mathbb{R}, \ f(x) \geq \alpha x + \beta, \forall x \in \mathbb{R} \}.
\]

It is well-known (see for example [26, Theorem 11.1]) that

\[
f^* = (\text{conv } f)^* = (f)^* = (\text{conv } f)^*.
\]

Moreover, if \( \text{conv } f \) is proper, \( f^{**} \) is also proper, convex and l.s.c. and

\[
f^{**} = \text{conv } f.
\]
Note that the concave envelop has also been used by [6] in order to compute the classical super-replication price under the no-arbitrage condition using the dual formulation.

In order to compute \( p(g) \), we need to compute \( \text{conv} f \) and \( \text{conv} g \). To do so, we introduce the notion of relative concave envelop of \( g \) with respect to \( \text{supp} H Y \):

\[
\text{conc}(g, \text{supp} H Y)(x) = \inf \{ v(x), \ v \text{ is concave and } v(z) \geq g(z), \forall z \in \text{supp} H Y \}.
\]

**Lemma 2.3.** Suppose that \( g \) is a \( H \)-normal integrand. Then, we have that if \( x \notin \text{convsupp} H Y \), \( \text{conv} f(x) = \text{conv} g(x) = +\infty \) and if \( x \in \text{convsupp} H Y \),

\[
\begin{align*}
\text{conv} f(x) &= -\text{conc}(g, \text{supp} H Y)(x) \\
\text{conv} g(x) &= -\text{conc}(g, \text{supp} H Y)(x) \\
&= -\inf \{ \alpha x + \beta, \alpha, \beta \in \mathbb{R}, \alpha z + \beta \geq g(z), \forall z \in \text{supp} H Y \},
\end{align*}
\]

where \( \text{convsupp} H Y \) is the convex envelop of \( \text{supp} H Y \), i.e. the smallest convex set that contains \( \text{supp} H Y \).

**Remark 2.4.** Note that \( \text{conv} f \) is proper if and only if \( \text{conc}(g, \text{supp} H Y)(x) < +\infty \) for all \( x \in \text{convsupp} H Y \), since \( \text{convsupp} H Y \) is non-empty. So \( \text{conv} f \) is proper if there exists some concave function \( \varphi \) such that \( g \leq \varphi \) on \( \text{supp} H Y \) and \( \varphi < \infty \) on \( \text{convsupp} H Y \) \(^1\) (by definition, \( \text{conc}(g, \text{supp} H Y) \leq \varphi \)). Since for all \( x \in \text{convsupp} H Y \), \( \text{conc}(g, \text{supp} H Y)(x) \geq g(x) > -\infty \), we get that \( \text{conc}(g, \text{supp} H Y)(x) \in \mathbb{R} \) and \( \text{conc}(g, \text{supp} H Y)(x) \in \mathbb{R} \). So one may write that

\[
\text{conv} f = -\text{conc}(g, \text{supp} H Y) + \delta_{\text{convsupp} H Y}, \quad \text{conv} g = -\text{conc}(g, \text{supp} H Y) + \delta_{\text{convsupp} H Y}.
\]

**Proof.** The convex envelop of \( f \) and the convex envelop of \( \text{supp} H Y \) can be written as follows (see [26, Proposition 2.31, Proposition 2.27, Theorem 2.29]):

\[
\begin{align*}
\text{conv} f(x) &= \inf \{ \sum_{i=1}^{n} \lambda_i f(x_i), \ n \geq 1, \ (\lambda_i)_{i \in \{1,...,n\}} \in \mathbb{R}^+_n, \ (x_i)_{i \in \{1,...,n\}} \in \mathbb{R}^n, \ x = \sum_{i=1}^{n} \lambda_i x_i, \ \sum_{i=1}^{n} \lambda_i = 1 \} \\
\text{convsupp} H Y &= \{ \sum_{i=1}^{n} \lambda_i x_i, \ n \geq 1, \ (\lambda_i)_{i \in \{1,...,n\}} \in \mathbb{R}^+_n, \ \sum_{i=1}^{n} \lambda_i = 1, \ x_i \in \text{supp} H Y \}.
\end{align*}
\]

\(^1\)This is equivalent to assume that there exists \( \alpha, \beta \in \mathbb{R} \), such that \( g(x) \leq \alpha x + \beta \) for all \( x \in \text{supp} H Y \).
Assume that $x \notin \text{convsupp}_H Y$. If $x = \sum_{i=1}^{n} \lambda_i x_i$ for some $n \geq 1$, $(\lambda_i)_{i \in \{1, \ldots, n\}} \in \mathbb{R}_+^n$, and $(x_i)_{i \in \{1, \ldots, n\}} \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, there exists at least one $x_i \notin \text{supp}_H Y$ and $f(x_i) = +\infty$ and also $\text{conv} f(x) = +\infty$.

If $x \in \text{convsupp}_H Y$, by definition $\text{conv} f(x) = -\text{conc}(g, \text{supp}_H Y)(x)$ and the last equality follows easily. □

We deduce the following representation of the infimum super-hedging cost:

**Proposition 2.5.** Suppose that $g$ is a $\mathcal{H}$-normal integrand and that there exists some concave function $\varphi$ such that $g \leq \varphi$ on $\text{supp}_H Y$ and $\varphi < \infty$ on $\text{convsupp}_H Y$. Then,

$$p(g) = -\text{conv} f(y) = \text{conc}(g, \text{supp}_H Y)(y) - \delta_{\text{convsupp}_H Y}(y) \quad \text{a.s.}$$

We see that the fact that $y$ belongs a.s. to $\text{convsupp}_H Y$ or not is important for the value of $p(g)$. In particular in some cases, the infimum super-hedging price of a European claim may be $-\infty$. This is related to the notion of absence of immediate profit that we present now. We say that there is an immediate profit when it is possible to super-replicate the contingent claim 0 at a negative super-hedging price $p$.

**Definition 2.6.** There is an immediate profit (IP) if $p(0) \leq 0$ with $P(p(0) < 0) > 0$. On the contrary case, we say that the Absence of Immediate Profit (AIP) condition holds if $p(0) = 0$ a.s.

Notice that the (AIP) condition may be seen as a particular case of the utility based No Good Deal condition introduced by Cherny, see [10, Definition 3]. We know propose several characterization of the (AIP) condition.

**Proposition 2.7.** (AIP) holds if and only if $y \in \text{convsupp}_H Y$ a.s. Moreover,

$$\text{convsupp}_H Y = [\text{ess inf}_H Y, \text{ess sup}_H Y] \cap \mathbb{R}. \quad (2.6)$$

**Proof.** The assumptions of Proposition 2.5 are satisfied for $g = 0$ and we get that $p(0) = -\delta_{\text{convsupp}_H Y}(y)$ a.s. Hence, (AIP) holds true if and only if $y \in \text{convsupp}_H Y$ a.s. Then (2.6) follows from Lemma 6.11. □

**Corollary 2.8.** The (AIP) condition holds true if and only if $p(g) \geq 0$ a.s. for some non-negative $\mathcal{H}$-normal integrand $g$ such that there exists some concave function $\varphi$ verifying that $g \leq \varphi < \infty$.

In particular, the (AIP) condition holds true if and only the infimum super-hedging cost of some European call option is non-negative.
Proof. Assume that (AIP) condition holds true. Then, from Definition 2.6, we get that \( p(0) = 0 \) a.s. As \( g \geq 0 \), it is clear that \( p(g) \geq p(0) = 0 \) a.s. Conversely, assume that there exists some (IP). From Proposition 2.5, we get that

\[
p(g) = \text{conc}(g, \text{supp}_H(y)) - \delta_{\text{convsupp}_H}(y).
\]

By Proposition 2.7, we deduce that \( P(y \in \text{convsupp}_H) < 1 \) and, since \( \text{conc}(g, \text{supp}_H(y)) \leq \varphi < \infty \), \( P(p(g) = -\infty) > 0 \) and the converse is proved.

Lemma 2.9. (AIP) holds true if and only \( \mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{H}) = \{0\} \).

Proof. Using Lemma 2.2, we get that

\[
\mathcal{P}(0) = \{\text{ess sup}_H (-\theta Y) + \theta y, \theta \in L^0(\mathbb{R}, \mathcal{H}) \} + L^0(\mathbb{R}_+, \mathcal{H})
\]

\[
= \{-\theta (1_{\theta < 0} \text{ess sup}_H (Y - y) + 1_{\theta \geq 0} \text{ess inf}_H (Y - y)), \theta \in L^0(\mathbb{R}, \mathcal{H})\} + L^0(\mathbb{R}_+, \mathcal{H}).
\]

So \( \mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{H}) = \{0\} \cup (\Omega \setminus A^+ \cap A^-) \), where \( A^+ = \{\text{ess sup}_H(Y - y) \geq 0\} \) and \( A^- = \{\text{ess inf}_H(Y - y) \leq 0\} \). Using Proposition 2.7, we have that (AIP) holds true if and only \( P(A^+ \cap A^-) = 1 \) and the conclusion follows.

We now compare the (AIP) condition with the classical No Arbitrage (NA) one, which definition is recalled below.

Definition 2.10. The No Arbitrage (NA) condition holds true if for \( \theta \in L^0(\mathbb{R}, \mathcal{H}) \), \( \theta(Y - y) \geq 0 \) a.s. implies that \( \theta(Y - y) = 0 \) a.s. or equivalently

\[
\{\theta(Y - y) - \epsilon^+, \theta \in L^0(\mathbb{R}, \mathcal{H}), \epsilon^+ \in L^0(\mathbb{R}_+, \mathcal{F})\} \cap L^0(\mathbb{R}_+, \mathcal{F}) = \{0\}
\]

or equivalently \( \mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}) = \{0\} \).

Lemma 2.11. The (AIP) condition is strictly weaker than the (NA) one.

Proof. It is clear from Lemma 2.9 and Definition 2.10 that (NA) implies (AIP). We now provide some examples where (AIP) holds true and is strictly weaker than (NA). Using the Fundamental Theorem of Asset Pricing (FTAP, see [12]) and Remark 6.5, this is the case if there exists \( Q_1, Q_2 \ll P \) such that \( S \) is a \( Q_1 \)-super martingale (resp. \( Q_2 \)-sub martingale). This is of course true if \( \text{ess inf}_H Y = 0 \) and \( \text{ess sup}_H Y = \infty \). Finally, this is also the case for a model of the form \( Y = yZ \) where \( Z > 0 \) is such that \( \text{supp}_H Z = [0, 1] \) a.s. or \( \text{supp}_H Z = [1, \infty) \) a.s. and \( y > 0 \). Indeed (recall Lemma 6.11), if \( \text{supp}_H Z = [0, 1] \), \( \text{ess inf}_H Y = y \text{ ess inf}_H Z = 0 < y \) and \( \text{ess sup}_H Y = y \text{ ess sup}_H Z = y \geq y \). The same holds if \( \text{supp}_H Z = [1, \infty) \) a.s.
Nevertheless, this kind of model does not admit a risk-neutral probability measure and the (NA) condition does not hold true using the FTAP. Indeed, in the contrary case, there exists a density process i.e. a positive martingale \((\rho_t)_{t \in \{0,1\}}\) with \(\rho_0 = 1\) such that \(\rho S\) is a \(P\)-martingale: \(\mathbb{E}^P(\rho_1 Y|\mathcal{H}) = \rho_0 y\).

We get that \(\mathbb{E}^P(\rho_1 Z|\mathcal{H}) = \rho_0\). Since we also have \(\rho_0 = \mathbb{E}^P(\rho_1|\mathcal{H})\), we deduce that \(\mathbb{E}^P(\rho_1 (1 - Z)|\mathcal{H}) = 0\). Since \(Z \leq 1\) a.s. or \(Z \geq 1\) a.s., this implies that \(\rho_1 (1 - Z) = 0\) hence \(Z = 1\) which yields a contradiction. \(\square\)

We finish the one-period study with the characterization of the infimum super-hedging cost under the (AIP) condition.

**Corollary 2.12.** Suppose that (AIP) holds true. Let \(g\) be a \(\mathcal{H}\)-normal integrand, such that there exists some concave function \(\varphi\) verifying that \(g \leq \varphi\) on \(\text{supp} \mathcal{H} Y\) and \(\varphi < \infty\) on \(\text{convsupp} \mathcal{H} Y\). Then, a.s.

\[
p(g) = \text{conc}(g, \text{supp} \mathcal{H} Y)(y) = \inf \{\alpha y + \beta, \alpha, \beta \in \mathbb{R}, \alpha x + \beta \geq g(x), \forall x \in \text{supp} \mathcal{H} Y\}. \tag{2.7}
\]

So, in the case where \(g\) is concave and u.s.c., we get under (AIP) that \(p(g) = g(y)\) a.s.

If \(g\) is convex and \(\lim_{x \to \infty} x^{-1}g(x) = M \in \mathbb{R}\), the relative concave envelop of \(g\) with respect to \(\text{supp} \mathcal{H} Y\) is the affine function that coincides with \(g\) on the extreme points of the interval \(\text{convsupp} \mathcal{H} Y\) i.e. a.s.

\[
p(g) = \theta^* y + \beta^* = g(\text{ess inf} \mathcal{H} Y) + \theta^* (y - \text{ess inf} \mathcal{H} Y), \tag{2.8}
\]

\[
\theta^* = \frac{g(\text{ess sup} \mathcal{H} Y) - g(\text{ess inf} \mathcal{H} Y)}{\text{ess sup} \mathcal{H} Y - \text{ess inf} \mathcal{H} Y}, \tag{2.9}
\]

where we use the conventions \(\theta^* = 0 = 0\) in the case \(\text{ess sup} \mathcal{H} Y = \text{ess inf} \mathcal{H} Y\) a.s. and \(\theta^* = \frac{g(\infty)}{\infty} = M\) if \(\text{ess inf} \mathcal{H} Y < \text{ess sup} \mathcal{H} Y = +\infty\) a.s. Moreover, using (2.7), we get that \(\theta^* Y + \beta^* \geq g(Y)\) a.s. (recall that \(Y \in \text{supp} \mathcal{H} Y\)) and this implies by (2.8) that

\[
p(g) + \theta^*(Y - y) \geq g \text{ a.s.} \tag{2.10}
\]

and \(p(g) \in \mathcal{P}(g)\).
3. The multi-period framework

3.1. Multi-period super-hedging prices

For every $t \in \{0, \ldots, T\}$ the set $\mathcal{R}_t^T$ of all claims that can be super-replicated from the zero initial endowment at time $t$ is defined by

$$\mathcal{R}_t^T := \left\{ \sum_{u=t+1}^{T} \theta_{u-1} \Delta S_u - \epsilon^+_T, \theta_{u-1} \in L^0(\mathbb{R}, \mathcal{F}_{u-1}), \epsilon^+_T \in L^0(\mathbb{R}_+, \mathcal{F}_T) \right\}.$$

(3.11)

The set of (multi-period) super-hedging prices and the (multi-period) infimum super-hedging cost of some contingent claim $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$ at time $t$ are given by for all $t \in \{0, \ldots, T-1\}$ by

$$\Pi_{t,T}(g_T) = \left\{ g_T \right\},$$
$$\Pi_{t,T}(g_T) = \left\{ x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists R \in \mathcal{R}_t^T, x_t + R = g_T \text{ a.s.} \right\}$$
$$\pi_{t,T}(g_T) = \text{ess inf}_{\mathcal{F}_t} \Pi_{t,T}(g_T).$$

(3.12)

As in the one-period case, it is clear that the infimum super-hedging cost is not necessarily a price in the sense that $\pi_{t,T}(g_T) \notin \Pi_{t,T}(g_T)$ when $\Pi_{t,T}(g_T)$ is not closed. Alternatively, we may define sequentially for all $t \in \{0, \ldots, T-1\}$

$$\mathcal{P}_{t,T}(g_T) = \{ g_T \}$$
$$\mathcal{P}_{t,T}(g_T) = \left\{ x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists p_{t+1} \in \mathcal{P}_{t+1,T}(g_T), x_t + \theta_t \Delta S_{t+1} \geq p_{t+1} \text{ a.s.} \right\}.$$

The set $\mathcal{P}_{t,T}(g_T)$ contains at time $t$ all the super-hedging prices for some price $p_{t+1} \in \mathcal{P}_{t+1,T}(g_T)$ at time $t+1$. First we show that for all $t \in \{0, \ldots, T\}$

$$\Pi_{t,T}(g_T) = \mathcal{P}_{t,T}(g_T).$$

(3.13)

It is clear at time $T$. Let $x_t \in \Pi_{t,T}(g_T)$. Then there exists for all $u \in \{t, \ldots, T-1\}, \theta_u \in L^0(\mathbb{R}, \mathcal{F}_u)$ such that $x_t + \sum_{u=t+1}^{T-1} \theta_{u-1} \Delta S_u + \theta_{T-1} \Delta S_T \geq g_T$ a.s. As $g_T \in \mathcal{P}_{T,T}(g_T)$,

$$x_t + \sum_{u=t+1}^{T-2} \theta_{u-1} \Delta S_u + \theta_{T-2} \Delta S_{T-1} = x_t + \sum_{u=t+1}^{T-1} \theta_{u-1} \Delta S_u \in \mathcal{P}_{T-1,T}(g_T)$$

and $x_t + \sum_{u=t+1}^{T-2} \theta_{u-1} \Delta S_u \in \mathcal{P}_{T-2,T}(g_T)$ and recursively $x_t \in \mathcal{P}_{t,T}(g_T)$. Conversely, let $x_t \in \mathcal{P}_{t,T}(g_T)$, then there exists $\theta_t \in L^0(\mathbb{R}, \mathcal{F}_t)$ and $p_{t+1} \in \mathcal{P}_{t+1,T}(g_T)$. Therefore, the infimum super-hedging cost is a price in the sense that $\pi_{t,T}(g_T) \in \Pi_{t,T}(g_T)$.\]
We now define a local version of super-hedging prices. Let \( g_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1}) \), then the set of one-step super-hedging prices of \( g_{t+1} \) and its associated infimum super-hedging cost are given by

\[
P_{t,t+1}(g_{t+1}) = \{ x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}, \mathcal{F}_t), x_t + \theta_t \Delta S_{t+1} \geq g_{t+1} \text{ a.s.} \}
\]

\[
\pi_{t,t+1}(g_{t+1}) = \text{ess inf}_\mathcal{F}_t P_{t,t+1}(g_{t+1}).
\]

The following lemma makes the link between local and global super-hedging under the assumption that the infimum (global) super-replication cost is a price. It also provides a dynamic programming principle.

**Lemma 3.1.** Let \( g_T \in L^0(\mathbb{R}, \mathcal{F}_T) \) and \( t \in \{0, \ldots, T-1\} \). Then \( P_{t,T}(g_T) \subset \mathcal{P}_{t+1,T}(\pi_{t+1,T}(g_T)) \) and \( \pi_{t,T}(g_T) \geq \pi_{t+1,T}(\pi_{t+1,T}(g_T)) \). Moreover if \( \pi_{t+1,T}(g_T) \in \Pi_{t+1,T}(g_T) \), then \( P_{t,T}(g_T) = \mathcal{P}_{t+1}(\pi_{t+1,T}(g_T)) \) and \( \pi_{t,T}(g_T) = \pi_{t+1}(\pi_{t+1,T}(g_T)) \).

**Remark 3.2.** We will give in Proposition 4.4 condition implying that \( \pi_{t+1,T}(g_T) \in \mathcal{P}_{t+1,T}(g_T) \). Under (AIP), if at each step, \( \pi_{t+1,T}(g_T) \in \Pi_{t+1,T}(g_T) \) and if \( \pi_{t+1,T}(g_T) = g_{t+1}(S_{t+1}) \) for some “nice” \( \mathcal{F}_t \)-normal integrand \( g_{t+1} \), we will get from Corollary 2.12 that \( \pi_{t,T}(g_T) = \text{ess inf}_t \mathcal{P}_{t+1}(S_{t+1}) \) a.s. We will propose in Section 5 a quite general setting where this holds true.

**Proof.** Let \( x_t \in \mathcal{P}_{t,T}(g_T) \), then there exists \( \theta_t \in L^0(\mathbb{R}, \mathcal{F}_t) \) and \( p_{t+1} \in \mathcal{P}_{t+1,T}(g_T) \) such that \( (\text{recall } (3.13)) \)

\[
x_t + \theta_t \Delta S_{t+1} \geq p_{t+1} \geq \text{ess inf}_\mathcal{F}_t \mathcal{P}_{t+1,T}(g_T) = \pi_{t+1,T}(g_T) \text{ a.s.}
\]

and the first statement follows. The second one follows directly from \( \pi_{t+1,T}(g_T) \in \mathcal{P}_{t+1,T}(g_T) \). \( \square \)

### 3.2. Multi-period (AIP)

We now define the notion of global and local immediate profit at time \( t \). The global one says that it is possible to super-replicate from a negative cost at time \( t \) the claim 0 payed at time \( T \) and the local one the claim 0 payed at time \( t+1 \). We will see that they are equivalent.
Definition 3.3. Fix \( t \in \{0, \ldots, T\} \). A global immediate profit (IP) at time \( t \) is a non-null element of \( \Pi_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) \). We say that (AIP) condition holds at time \( t \) if there is no global IP at \( t \), i.e. if \( \Pi_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\} \). A local immediate profit (LIP) at time \( t \) is a a non-null element of \( \mathcal{P}_{t,t+1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) \). We say that (ALIP) condition holds at time \( t \) if there is no local IP at \( t \), i.e. if \( \mathcal{P}_{t,t+1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\} \). Finally we say that the (AIP) condition holds true if the (AIP) condition holds at time \( t \) for all \( t \in \{0, \ldots, T\} \).

Using Proposition 2.7, we get the equivalence between the (ALIP) condition at time \( t \) and the fact that \( S_t \in \text{convsupp}_t S_{t+1} \) a.s. So Theorem 3.4 below will show that there is an equivalence between (ALIP) at time \( t \) and (AIP) at time \( t \).

Theorem 3.4. (AIP) holds if and only if one of the the following assertions holds:

1. \( S_t \in \text{convsupp}_t S_{t+1} \) a.s., for all \( t \in \{0, \ldots, T-1\} \).
2. \( \text{ess inf}_t S_{t+1} \leq S_t \leq \text{ess sup}_t S_{t+1} \) a.s., for all \( t \in \{0, \ldots, T-1\} \).
3. \( \text{ess inf}_t S_u \leq S_t \leq \text{ess sup}_t S_u \) a.s., for all \( u \in \{t, \ldots, T\} \).
4. \( \pi_{t,T}(0) = 0 \) a.s. for all \( t \in \{0, \ldots, T-1\} \).

Proof. Let \( A_T = \Omega \) and for all \( t \in \{0, \ldots, T-1\} \),

\[
A_t := \{\text{ess sup}_t \Delta S_{t+1} \geq 0\} \cap \{\text{ess inf}_t \Delta S_{t+1} \leq 0\}.
\]

At time \( T \), \( \mathcal{P}_{T,T}(0) = \{0\} \), thus (AIP) holds at \( T \) and \( \pi_{T,T}(0) = 0 \). We show by induction that \( 0 \in \mathcal{P}_{t,T}(0) \) and that under (AIP) at time \( t + 1 \)

\[
\pi_{t,T}(0) = 0 \text{ a.s. } \iff P(A_t) = 1 \iff (AIP) \text{ holds at time } t.
\]

As (AIP) is equivalent to (AIP) at time \( t \) for all \( t \in \{0, \ldots, T\} \), this proves 1., 2. and 4. (recall (2.6)). Assertion 3. follows from Lemma 6.6. We proceed by backward recursion. Consider \( t \in \{0, \ldots, T-1\} \), assume that the induction hypothesis holds true at \( t + 1 \) and that (AIP) holds at time \( t + 1 \). The proof is very similar to the one of Lemma 2.9. As \( \pi_{t+1,T}(0) = 0 \in \mathcal{P}_{t+1,T}(0) \), we can apply Lemmata 3.1 and 2.2 and

\[
\mathcal{P}_{t,T}(0) = \mathcal{P}_{t,t+1}(0) = \{\text{ess sup}_t (-\theta S_{t+1} + \theta S_t) \in L^0(\mathbb{R}, \mathcal{F}_t) \} + L^0(\mathbb{R}_+, \mathcal{F}_t)
\]

\[
= \{-\theta \{1_{\theta < 0} \text{ess sup}_t \Delta S_{t+1} + 1_{\theta \geq 0} \text{ess inf}_t \Delta S_{t+1}\} \in L^0(\mathbb{R}_+, \mathcal{F}_t) \} + L^0(\mathbb{R}_+, \mathcal{F}_t).
\]
Note that $0 \in \mathcal{P}_{t,T}(0)$. Moreover, (AIP) holds at time $t$ if and only if $P(A_t) = 1$ (this is also a direct consequence of Proposition 2.7). We also obtain that $\pi_{t,T}(0) = \text{ess inf}_t \mathcal{P}_{t,T}(0)$ is equal to 0 on $A_t$ and $-\infty$ on $\Omega \setminus A_t$. So (AIP) holds at time $t$ if and only if $\pi_{t,T}(0) = 0$ a.s. In particular, under (AIP) at time $t$, the infimum super-hedging cost of 0 at time $t$ is $\pi_{t,T}(0) = 0 \in \mathcal{P}_{t,T}(0)$.

4. Comparison between the (AIP) condition and classical no-arbitrage conditions

The goal of this section is to compare the (AIP) condition with different definitions of no-arbitrage and no-free lunch. Recall that the set of all super-hedging prices for the zero claim at time $t$ is given by $\Pi_{t,T}(0) = (-\mathcal{R}_T^t) \cap L^0(\mathbb{R}, \mathcal{F}_t)$ (see (3.11) and (3.12)). It follows that (AIP) reads as $R_T^t \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$, for all $t \in \{0, \ldots, T\}$ (see Definition 3.3). We first recall the multiperiod no-arbitrage (NA) condition.

**Definition 4.1.** The no arbitrage (NA) condition holds true if $R_T^t \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$ for all $t \in \{0, \ldots, T\}$.

It is easy to see that the (NA) condition can also be stated as follows: $V_{T,\theta}^0 \geq 0$ a.s. implies that $V_{T,\theta}^0 = 0$ a.s. The (AIP) condition is strictly weaker than the (NA) one. It is clear that the (NA) condition implies the (AIP) one and one can adapt the counter-examples of Lemma 2.11 to get that the equivalence does not hold true. We will see in Lemma 4.6 that the (AIP) condition is not necessarily equivalent to a stronger condition than (AIP) that we call (AWIP) (for absence of weak immediate profit):

**Definition 4.2.** The absence of weak immediate profit (AWIP) condition holds true if $\overline{R}_T^t \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$ for all $t \in \{0, \ldots, T\}$, where the closure of $R_T^t$ is taken with respect to the convergence in probability.

Note that this condition is a weak form of the classical No Free Lunch condition $\overline{R}_T^t \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$ for all $t \in \{0, \ldots, T\}$. The following result implies that (AWIP) may be equivalent to (AIP) condition under an extra closeness condition. It also provides a characterization through (absolutely continuous) martingale measures.

**Theorem 4.3.** The following statements are equivalent:

- (AWIP) holds.
For every $t \in \{0, \ldots, T\}$, there exists $Q \ll P$ with $\mathbb{E}(dQ/dP|\mathcal{F}_t) = 1$ such that $(S_u)_{u \in \{t, \ldots, T\}}$ is a $Q$-martingale.

(AIP) holds and $\overline{\mathcal{R}}^T_t \cap L^0(\mathbb{R}, \mathcal{F}_t) = \mathcal{R}^T_t \cap L^0(\mathbb{R}, \mathcal{F}_t)$ for every $t \in \{0, \ldots, T\}$.

Proof. Suppose that (AWIP) holds and fix some $t \in \{0, \ldots, T\}$. We may suppose without loss of generality that the process $S$ is integrable under $P$. Under (AWIP), we then have $\mathcal{R}^T_t \cap L^1(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$ where the closure is taken in $L^1$. Therefore, for every nonzero $x \in L^1(\mathbb{R}_+, \mathcal{F}_t)$, there exists by the Hahn-Banach theorem a non-zero $\mathcal{L}$ such that $\mathcal{L}(x) = 0$. Suppose that for every $\gamma \in \mathcal{L}$, we deduce that $\mathcal{L}(x) = 0$. By [19, Lemma 2.1.3], we deduce an at most countable subfamily $(u_i)_{i \geq 1}$ such that the union $\bigcup_i \{\mathbb{E}(x_i|\mathcal{F}_t) > 0\}$ is of full measure. Therefore, $Z = \sum_{i=1}^{\infty} 2^{-i} Z_{x_i} \geq 0$ is such that $\mathbb{E}(Z|\mathcal{F}_t) > 0$ and we define $Q \ll P$ such that $dQ = (Z/\mathbb{E}(Z|\mathcal{F}_t))dP$. As the subset $\{\sum_{u=t+1}^{T} \theta_{u-1} \Delta S_u, \theta_{u-1} \in L(\mathbb{R}, \mathcal{F}_{u-1})\}$ is a linear vector space contained in $\mathcal{R}^T_t$, we deduce that $(S_u)_{u \in \{t, \ldots, T\}}$ is a $Q$-martingale.

Suppose that for every $t \in \{0, \ldots, T\}$, there exists $Q \ll P$ such that $(S_u)_{u \in \{t, \ldots, T\}}$ is a $Q$-martingale with $\mathbb{E}(dQ/dP|\mathcal{F}_t) = 1$. Let us define for $u \in \{t, \ldots, T\}$, $\rho_u = \mathbb{E}_P(dQ/dP|\mathcal{F}_u)$ then $\rho_u \geq 0$ and $\rho_t = 1$. Consider $\gamma_t \in \mathcal{R}^T_t \cap L^0(\mathbb{R}_+, \mathcal{F}_t)$, i.e., $\gamma_t$ is $\mathcal{F}_t$-measurable and is of the form $\gamma_t = \sum_{u=t}^{T-1} \theta_u \Delta S_{u+1} - \epsilon^+_t$. Since $\theta_u$ is $\mathcal{F}_u$-measurable, $\theta_u \Delta S_{u+1}$ admits a generalized conditional expectation under $Q$ knowing $\mathcal{F}_u$ and, by assumption, we have $\mathbb{E}_Q(\theta_u \Delta S_{u+1}|\mathcal{F}_u) = 0$. We deduce by the tower law that a.s.

$$
\gamma_t = \mathbb{E}_Q(\gamma_t|\mathcal{F}_t) = \sum_{u=t}^{T-1} \mathbb{E}_Q(\mathbb{E}_Q(\theta_u \Delta S_{u+1}|\mathcal{F}_u)|\mathcal{F}_t) - \mathbb{E}_Q(\epsilon^+_t|\mathcal{F}_t) = -\mathbb{E}_Q(\epsilon^+_t|\mathcal{F}_t).
$$

Hence $\gamma_t = 0$ a.s., i.e. (AIP) holds. It remains to show that $\overline{\mathcal{R}}^T_t \cap L^0(\mathbb{R}, \mathcal{F}_t) \subseteq \mathcal{R}^T_t \cap L^0(\mathbb{R}, \mathcal{F}_t)$.

Consider first a one step model, where $(S_u)_{u \in \{T-1, T\}}$ is a $Q$-martingale with $\rho_T \geq 0$ and $\rho_{T-1} = 1$. Suppose that $\gamma^n = \theta^n_{T-1} \Delta S_T - \epsilon^n_{T+} \in L^0(\mathbb{R}, \mathcal{F}_T)$ converges in probability to $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_{T-1})$. We need to show that $\gamma^\infty \in \mathcal{R}^T_{T-1}$. On the $\mathcal{F}_{T-1}$-measurable set $\Lambda_{T-1} := \{\liminf_n |\theta^n_{T-1}| < \infty\}$, by [19, Lemma
2.1.2], we may assume w.l.o.g. that \( \theta^n_{T-1} \) is convergent to some \( \theta^n_{\infty} \) hence \( \epsilon^n_{T-1} \) is also convergent and we can conclude. Otherwise, on \( \Omega \setminus \Lambda_{T-1} \), we use the normalized sequences \( \hat{\theta}^n_{T-1} := \theta^n_{T-1}/(\theta^n_{T-1} + 1) \), \( \bar{\epsilon}^n_{T} := \epsilon^n_{T}/(\theta^n_{T-1} + 1) \).

By [19, Lemma 2.1.2], we may assume that a.s. \( \theta^n_{T-1} \to \bar{\theta}_{T-1}^\infty \), \( \bar{\epsilon}^n_{T} \to \bar{\epsilon}^\infty_{T} \) and \( \bar{\theta}_{T-1}^\infty \Delta S_T - \bar{\epsilon}^\infty_{T} = 0 \) a.s. As \( |\bar{\theta}_{T-1}^\infty| = 1 \) a.s., first consider the subset \( \Lambda^2_{T-1} := (\Omega \setminus \Lambda_{T-1}) \cap \{ \hat{\theta}_{T-1}^\infty = 1 \} \) and \( \Delta S_T \geq 0 \) a.s. on \( \Lambda^2_{T-1} \). Since \( \mathbb{E}_Q(\Delta S_T 1_{\Lambda^2_{T-1}} | \mathcal{F}_{T-1}) = 0 \) a.s., we get that \( \rho_T \Delta S_T 1_{\Lambda^2_{T-1}} = 0 \) a.s. Hence \( \rho_T \gamma^n_{\Lambda^2_{T-1}} = -\rho_T \epsilon^n_{T} 1_{\Lambda^2_{T-1}} \leq 0 \) a.s. Taking the limit, we get that \( \rho_T \gamma^\infty_{\Lambda^2_{T-1}} \leq 0 \) a.s. and, since \( \gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_{T-1}) \), we deduce that \( \rho_T \gamma^\infty_{\Lambda^2_{T-1}} \leq 0 \) a.s. Recall that \( \rho_T = 1 \) hence \( \gamma^\infty_{\Lambda^2_{T-1}} \leq 0 \) a.s. and \( \gamma^\infty_{\Lambda^2_{T-1}} \in R^T_{T-1} \). On the subset \( \{ \Omega \setminus \Lambda_{T-1} \} \cap \{ \hat{\theta}_{T-1}^\infty = -1 \} \) we may argue similarly and the conclusion follows in the one step model.

Fix some \( s \in \{ t, \ldots, T - 1 \} \). We show that \( R^T_{s+1} \cap L^0(\mathbb{R}, \mathcal{F}_{s+1}) \subseteq R^T_{s+1} \cap L^0(\mathbb{R}, \mathcal{F}_{s+1}) \) implies the same property for \( s \) instead of \( s + 1 \). By assumption \( (S_u)_{u \in \{ s, \ldots, T \}} \) is a Q-martingale and \( \mathbb{E}_P(dQ/dP | \mathcal{F}_u) = \rho_u \geq 0 \) for \( u \in \{ s, \ldots, T \} \) and \( \rho_s = 1 \). Suppose that \( \gamma^n = \sum_{u=s}^{T-1} \theta^n_u \Delta S_{u+1} - \bar{\epsilon}^n_{u+1} \in L^0(\mathbb{R}, \mathcal{F}_T) \) converges to \( \gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_T) \). If \( \gamma^\infty = 0 \) there is nothing to prove. On the \( \mathcal{F}_s \) measurable set \( \Lambda_s := \{ \liminf_n |\theta^n_s| < \infty \} \), by [19, Lemma 2.1.2], we may assume w.l.o.g. that \( \theta^n_s \) converges to \( \theta^\infty \). Therefore, by the induction hypothesis, \( \sum_{u=s+1}^{T-1} \theta^n_u \Delta S_{u+1} - \bar{\epsilon}^n_{u+1} \) is also convergent to an element of \( R^T_{s+1} \cap L^0(\mathbb{R}, \mathcal{F}_{s+1}) \) and we conclude that \( \gamma^\infty \in R^T_s \). On \( \Omega \setminus \Lambda_{s-1} \), we use the normalisation procedure as before, and deduce the equality \( \sum_{u=s}^{T-1} \bar{\theta}^\infty_u \Delta S_{u+1} - \bar{\epsilon}^\infty_{u+1} = 0 \) a.s. for some \( \bar{\theta}^\infty_u \in L^0(\mathbb{R}, \mathcal{F}_u) \), \( u \in \{ s, \ldots, T - 1 \} \) and \( \bar{\epsilon}^\infty_{u+1} \) such that \( |\bar{\theta}^\infty_s| = 1 \) a.s. We then argue on \( \Lambda^2_s := (\Omega \setminus \Lambda_{s-1}) \cap \{ \bar{\theta}^\infty_s = 1 \} \) and \( \Lambda^3_s := (\Omega \setminus \Lambda_{s-1}) \cap \{ \bar{\theta}^\infty_s = -1 \} \) respectively. When \( \bar{\theta}^\infty_s = 1 \), we deduce that \( \Delta S_{s+1} + \sum_{u=s+1}^{T-1} \bar{\theta}^\infty_u \Delta S_{u+1} - \bar{\epsilon}^\infty_{u+1} = 0 \) a.s., i.e. \( \Delta S_{s+1} \in \Pi_{s+1,T}(0) \) hence \( \Delta S_{s+1} \geq \tau_{s+1,T}(0) = 0 \) a.s. under (AIP), see Theorem 3.4. Since \( \mathbb{E}_Q(\Delta S_{s+1} 1_{\Lambda^3_s} | \mathcal{F}_s) = 0 \) a.s., \( \rho_{s+1} \Delta S_{s+1} 1_{\Lambda^3_s} = 0 \) a.s. So, \( \rho_{s+1} \gamma^n 1_{\Lambda^3_s} \in R^T_{s+1} \) hence \( \rho_{s+1} \gamma^\infty 1_{\Lambda^3_s} \in R^T_{s+1} \cap L^0(\mathbb{R}, \mathcal{F}_{s+1}) \) by induction. As \( \rho_{s+1} \gamma^\infty 1_{\Lambda^3_s} \) admits a generalized conditional expectation knowing \( \mathcal{F}_s \), we deduce from (AIP) that \( \mathbb{E}_Q(\rho_{s+1} \gamma^\infty 1_{\Lambda^3_s} | \mathcal{F}_s) \leq 0 \) a.s. hence \( \rho_{s+1} \gamma^\infty 1_{\Lambda^3_s} \leq 0 \) a.s. Recall that \( \rho_s = 1 \) hence \( \gamma^\infty 1_{\Lambda^3_s} \leq 0 \) a.s. so that \( \gamma^\infty 1_{\Lambda^3_s} \in R^T_s \cap L^0(\mathbb{R}, \mathcal{F}_s) \).

Finally, notice that the (AIP) condition implies (AWIP) as soon as the equality \( R^T_t \cap L^0(\mathbb{R}+, \mathcal{F}_t) = R^T_t \cap L^0(\mathbb{R}+, \mathcal{F}_t) \) holds for every \( t \in \{ 0, \ldots, T - 1 \} \).

\[\square\]

**Proposition 4.4.** Suppose that \( P(\text{ess inf}_{\mathcal{F}_t} S_{t+1} = S_t) = P(\text{ess sup}_{\mathcal{F}_t} S_{t+1} = \ldots)\)
and, under these equivalent conditions, $\mathcal{R}_t^T$ is closed in probability for every $t \in \{0, \ldots, T-1\}$.

**Remark 4.5.** Under the assumption of Proposition 4.4, the infimum super-hedging cost is a super-hedging price.

**Proof.** From Theorem 4.3, it suffices to show that $\mathcal{R}_t^T$ is closed in probability for every $t \in \{0, \ldots, T-1\}$ under (AIP). Consider first the one step model, i.e. suppose that $\gamma^n = \theta^n_{t-1} \Delta S_T - \epsilon^n_T + \in \mathcal{R}_{t-1}^T$ is a convergent sequence to $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_T)$. Using [19, Lemma 2.1.2], it is sufficient to show that the $\mathcal{F}_{t-1}$-measurable set $\Lambda_{t-1} := \{ \liminf_n |\theta^n_{t-1}| < \infty \}$ satisfies $P(\Lambda_{t-1}) = 1$. Following the Theorem 4.3 proof’s normalization procedure on $\Omega \setminus \Lambda_{t-1}$, we get that $\hat{\theta}^\infty_{t-1} \Delta S_T \geq 0$ a.s. where $|\hat{\theta}^\infty_{t-1}| = 1$ a.s. First consider the subset $\Lambda^2_{t-1} := (\Omega \setminus \Lambda_{t-1}) \cap \{ \hat{\theta}^\infty_{t-1} = 1 \} \in \mathcal{F}_{t-1}$. We have $\Delta S_T \geq 0$ a.s. hence $\epsilon \inf_{\mathcal{F}_{t-1}} S_T \geq S_{t-1}$ a.s. on $\Lambda^2_{t-1}$. By (AIP) (see Theorem 3.4), we deduce that $\epsilon \sup_{\mathcal{F}_{t-1}} S_T = S_{t-1}$ a.s. on $\Lambda^2_{t-1}$. The assumption implies that $P(\Lambda^2_{t-1}) = 0$. On the remaining subset $\Lambda^3_{t-1} := (\Omega \setminus \Lambda_{t-1}) \cap \{ \hat{\theta}^\infty_{t-1} = -1 \} \in \mathcal{F}_{t-1}$, we obtain similarly that $\epsilon \sup_{\mathcal{F}_{t-1}} S_T = S_{t-1}$ a.s. and thus $P(\Lambda^3_{t-1}) = 0$.

By induction, assume that $\mathcal{R}_t^T$ is closed in probability and let us show that $\mathcal{R}_{t+1}^T$ is also closed in probability. To do so, suppose that $\gamma^n = \sum_{u=t+1}^T \theta^n_{u-1} \Delta S_u - \epsilon^n_T \in \mathcal{R}_t^T$ converges to $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_T)$. Again, it is enough to prove that $P(\Lambda_t) = 1$ with $\Lambda_t := \{ \liminf_u |\theta^n_u| < \infty \} \in \mathcal{F}_t$. Indeed on $\Lambda_t$, by [19, Lemma 2.1.2], we may assume w.l.o.g. that $\theta^n_u$ converges to $\theta^\infty_u$. By the induction hypothesis, we deduce that $\sum_{u=t+2}^T \theta^n_{u-1} \Delta S_u - \epsilon^n_T$ is also convergent to an element of $\mathcal{R}_t^T$ and $\gamma^\infty \in \mathcal{R}_t^T$. Using again the normalization procedure on $\Omega \setminus \Lambda_t$, we deduce that $\sum_{u=t+1}^T \theta^\infty_{u-1} \Delta S_u - \epsilon^\infty_T = 0$ a.s. where $\hat{\theta}^\infty_{u-1} \in L^0(\mathbb{R}, \mathcal{F}_{u-1})$, $u \in \{t, \ldots, T-1\}$ and $\epsilon^\infty_T \geq 0$ such that $\hat{\theta}^\infty_1 = 1$ a.s. We then argue on $\Lambda^2_t := (\Omega \setminus \Lambda_t) \cap \{ \hat{\theta}^\infty_t = 1 \} \in \mathcal{F}_t$ and $\Lambda^3_t := (\Omega \setminus \Lambda_t) \cap \{ \hat{\theta}^\infty_t = -1 \} \in \mathcal{F}_t$ respectively. On $\Lambda^2_t$, we obtain that $\Delta S_t \geq 0$ a.s. and $\epsilon \inf_{\mathcal{F}_t} S_{t+1} = S_t$ a.s. on $\Lambda^2_t$. This implies that $P(\Lambda^2_t) = 0$ and similarly $P(\Lambda^3_t) = 0$. The conclusion follows. $\square$

**Lemma 4.6.** The (AIP) condition is not necessarily equivalent to (AWIP).

**Proof.** Let us consider a positive process $(\tilde{S}_t)_{t \in \{0, \ldots, T\}}$ which is a $P$-martingale. We suppose that $\epsilon \inf_{\mathcal{F}_0} \tilde{S}_1 < \tilde{S}_1$ a.s., which holds in particular if $\tilde{S}$ a-
ometric Brownian motion as \( \text{ess inf}_{\mathcal{F}_0} \tilde{S}_1 = 0 \) a.s. Let us define \( S_t := \tilde{S}_t \) for \( t \in \{1, \ldots, T\} \) and \( S_0 := \text{ess inf}_{\mathcal{F}_0} S_1 \). We have \( \text{ess inf}_{\mathcal{F}_0} S_1 \leq S_0 \) and \( \text{ess sup}_{\mathcal{F}_0} S_1 \geq S_1 \geq \text{ess inf}_{\mathcal{F}_0} S_1 = S_0 \) hence (AIP) holds at time 0 (see Theorem 3.4). Moreover, by the martingale property, (AIP) also holds at any time \( t \in \{1, \ldots, T\} \) (see Remark 6.5). Let us suppose that (AWIP) holds. Using Theorem 4.3, there exists \( \rho_T \geq 0 \) with \( \mathbb{E}(\rho_T) = 1 \) such that \( S \) is a \( Q \)-martingale where \( dQ = \rho_T dP \). Therefore, \( \mathbb{E}(\rho_T \Delta S_1) = 0 \). Since \( \Delta S_1 > 0 \) by assumption, we deduce that \( \rho_T = 0 \) hence a contradiction. \( \square \)

5. Explicit pricing of a convex payoff under (AIP)

The aim of this section is to obtain some results in a particular model where \( \text{ess inf}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^d S_{t-1} \) a.s. and \( \text{ess sup}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^u S_{t-1} \) a.s. for every \( t \in \{1, \cdots, T\} \) with \( (k_{t-1,t}^d)_{t \in \{1, \cdots, T\}}, (k_{t-1,t}^u)_{t \in \{1, \cdots, T\}} \) and \( S_0 \) are deterministic non-negative numbers. We obtain the same computative scheme (see (5.14)) as in [9] but assuming only (AIP) and not (NA). We also propose some numerical experiments.

5.1. The algorithm

Theorem 5.1. Suppose that the model is defined by \( \text{ess inf}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^d S_{t-1} \) a.s. and \( \text{ess sup}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^u S_{t-1} \) a.s. where \( (k_{t-1,t}^d)_{t \in \{1, \cdots, T\}}, (k_{t-1,t}^u)_{t \in \{1, \cdots, T\}} \) and \( S_0 \) are deterministic non-negative numbers.

- The (AIP) condition holds at every instant \( t \) if and only if \( k_{t-1,t}^d \in [0, 1] \) and \( k_{t-1,t}^u \in [1, +\infty] \) for all \( t \in \{1, \cdots, T\} \).
- Suppose that the (AIP) condition holds. If \( h : \mathbb{R} \to \mathbb{R} \) is a non-negative convex function with \( \text{Dom} h = \mathbb{R} \) such that \( \lim_{z \to +\infty} \frac{h(z)}{z} \in [0, \infty) \), then the infimum super-hedging cost of the European contingent claim \( h(S_T) \) is a price and it is given by \( \pi_{t,T}(h) = h(t, S_t) \in \mathcal{P}_{t,T}(h(S_T)) \) a.s. where

\[
\begin{align*}
    &h(T, x) = h(x) \\
    &h(t - 1, x) = \lambda_{t-1,t} h(t, k_{t-1,t}^d x) + (1 - \lambda_{t-1,t}) h(t, k_{t-1,t}^u x), \\
\end{align*}
\]

(5.14)

where \( \lambda_{t-1,t} = \frac{k_{t-1,t}^u - k_{t-1,t}^d}{k_{t-1,t}^u - k_{t-1,t}^d} \in [0, 1] \) and \( 1 - \lambda_{t-1,t} = \frac{1 - k_{t-1,t}^d}{k_{t-1,t}^u - k_{t-1,t}^d} \in [0, 1] \), with the following conventions. When \( k_{t-1,t}^d = k_{t-1,t}^u = 1 \) or \( S_{t-1} = 0 \),
we distinguish three cases. If either
\[ S \] may continue the recursion as soon as
\[ P \]
\[ \pi_k = M \]
\[ t \]
multiply for every \( t \), consequence for convex functions (see (2.8) and (2.9)) and we get that a.s. \( \theta \)
\[ x \]
\[ 0.2 \]
given and, this result is illustrated through a numerical experiment in Section 5.2.

Proof. The conditions \( k_{-1,t}^d \in [0, 1] \) and \( k_{-1,t}^u \in [1, +\infty] \) for all \( t \in \{1, \ldots, T\} \) are equivalent to the (AIP) conditions by Theorem 3.4. We denote \( M = \frac{h(\infty)}{\infty} \) and \( M_t = \lim_{z \to +\infty} \frac{h(t, z)}{z} \). We prove the second statement. Assume that (AIP) holds true. We establish the recursive formulation \( \pi_{t,T}(h(S_t)) = h(t, S_t) \) given by (5.14), that \( h(t, \cdot) \geq h(t + 1, \cdot) \) and that \( M_t = M_{t+1} \). The case \( t = T \) is immediate. As \( h : R \to R \) is a convex function with Dom \( h = R \), \( h \) is clearly a \( \mathcal{F}_{T-1} \)-normal integrand, we can apply Proposition 2.5 and its consequence for convex functions (see (2.8) and (2.9)) and we get that a.s.

\[ \pi_{T-1,T}(h(S_T)) = h(k_{T-1,T}^d S_{T-1}^T) + \theta_{T-1}^* (S_{T-1} - k_{T-1,T}^d S_{T-1}) \]
\[ \theta_{T-1}^* = \frac{h(k_{T-1,T}^d S_{T-1}) - h(k_{T-1,T}^d S_{T-1})}{k_{T-1,T}^d S_{T-1} - k_{T-1,T}^d S_{T-1}} \]

where we use the conventions \( \theta_{T-1}^* = 0 \) if either \( S_{T-1} = 0 \) or \( k_{T-1,T}^d = k_{-1,T}^d = 1 \) and \( \theta_{T-1}^* = \frac{h(\infty)}{\infty} = M \) if \( k_{T-1,T}^d < k_{T-1,T}^u = +\infty \). Moreover, using (2.10), we obtain that \( \pi_{T-1,T}(h(S_t)) + \theta_{T-1}^* \Delta S_T \geq h(S_T) \) a.s. i.e. \( \pi_{T-1,T}(h(S_T)) \in \mathcal{P}(h(S_T)) \). So, using Lemma 3.1, we get that \( \mathcal{P}_{T-2,T}(h(S_T)) = \mathcal{P}_{T-2,T}(\pi_{T-1,T}(h(S_T))) \), \( \pi_{T-2,T}(h(S_T)) = \pi_{T-2,T}(\pi_{T-1,T}(h(S_T))) \) and we may continue the recursion as soon as \( \pi_{T-1,T}(h(S_T)) = h(T - 1, S_{T-1}) \) where \( h(T - 1, z) = 0 \) for all \( z \geq 0 \) and \( M_{T-1} = M \in [0, \infty) \). To see that we distinguish three cases. If either \( S_{T-1} = 0 \) or \( k_{T-1,T}^d = k_{T-1,T}^d = 1 \), \( \pi_{T-1,T}(h(S_T)) = h(S_{T-1}) \) and \( h(T - 1, z) = h(z) = h(T, z) \) satisfies all the

\[ \lambda_{t-1} = \frac{0}{0} = 0 \] and \( 1 - \lambda_{t-1} = 1 \) and when \( k_{t-1,t}^d < k_{t-1,t}^u = \infty \),
\[ \lambda_{t-1} = \frac{\infty}{\infty} = 1 \]
\[ (1 - \lambda_{t-1})h(t, (+\infty)x) = (1 - k_{t-1,t}^d)x \frac{h(t, (+\infty)x)}{(+\infty)x} \]
\[ = (1 - k_{t-1,t}^d)x \lim_{z \to +\infty} \frac{h(z)}{z} \].

Moreover, for every \( t \), \( \lim_{z \to +\infty} \frac{h(z)}{z} = \lim_{z \to +\infty} \frac{h(t, z)}{z} \) and \( h(\cdot, x) \) is non-increasing for all \( x \geq 0 \).
required conditions. If \( k^d_{T-1,T} < k^u_{T-1,T} = +\infty \), \( \pi_{T-1,T}(h) = h(k^d_{T-1,T}S_{T-1}) + M \), \( S_{T-1} - k^d_{T-1,T}S_{T-1} = h(T-1, S_{T-1}) \) with

\[
h(T-1, z) = h(k^d_{T-1,T}z) + M z \left( 1 - k^d_{T-1,T} \right)
\]

using (5.15). The term in the r.h.s. above is larger than \( h(z) = h(T, z) \) by convexity since \( \frac{k^u_{T-1,T} - 1}{k^u_{T-1,T} - k^d_{T-1,T}} h(k^d_{T-1,T}z) + \frac{1 - k^d_{T-1,T}}{k^u_{T-1,T} - k^d_{T-1,T}} h(k^u_{T-1,T}) \). As \( k^d_{T-1,T} \in [0, 1] \) and \( M \in [0, \infty) \), \( h(T-1, z) \geq 0 \) for all \( z \geq 0 \), we get that \( h(T-1, \cdot) \) is convex function with domain equal to \( \mathbb{R} \) since \( h \) is so. The function \( h(T-1, \cdot) \) also satisfies (5.14) (see (5.15)). Finally \( M_{T-1} = \lim_{z \to +\infty} k^d_{T-1,T} \frac{h(k^d_{T-1,T}z)}{k^u_{T-1,T}} + M \left( 1 - k^d_{T-1,T} \right) = M \).

The last case is when \( S_{T-1} \neq 0 \) and \( k^u_{T-1,T} \neq k^d_{T-1,T} \) and \( k^u_{T-1,T} < +\infty \). It is clear that (5.16) implies (5.14). Moreover as \( k^d_{T-1,T} \in [0, 1] \) and \( k^u_{T-1,T} \in [1, +\infty) \), \( \lambda_{T-1} = \frac{k^d_{T-1,T} - 1}{k^u_{T-1,T} - k^d_{T-1,T}} \in [0, 1] \) and \( 1 - \lambda_{T-1} = \frac{1 - k^d_{T-1,T}}{k^u_{T-1,T} - k^d_{T-1,T}} \in [0, 1] \) and (5.14) implies that \( h(T-1, z) \geq 0 \) for all \( z \geq 0 \), \( h(T-1, \cdot) \) is convex with domain equal to \( \mathbb{R} \) since \( h \) is so. Moreover,

\[
M_{T-1} = \lambda_{T-1} k^d_{T-1,T} \lim_{z \to +\infty} \frac{h(k^d_{T-1,T}z)}{k^d_{T-1,T}z} + \left( 1 - \lambda_{T-1} \right) k^u_{T-1,T} \lim_{z \to +\infty} \frac{h(k^u_{T-1,T}z)}{k^u_{T-1,T}z} = M,
\]

since

\[
\lambda_{T-1} k^d_{T-1,T} + \left( 1 - \lambda_{T-1} \right) k^u_{T-1,T} = 1.
\]

\( \square \)

**Remark 5.2.** The infimum super-hedging cost of the European contingent claim \( h(S_T) \) in our model is a price, precisely the same than the price we get in a binomial model \( S_t \in \{ k^d_{t-1,t} S_{t-1}, k^u_{t-1,t} S_{t-1} \} \) a.s., \( t = 1, \ldots, T \). Moreover, as in Corollary 2.8, one can prove that the (AIP) condition holds at every instant \( t \) if and only if the super-hedging prices of some European call option are non-negative.
5.2. Numerical experiments

5.2.1. Calibration

In this section, we suppose that the discrete dates are given by \( t^n_i = \frac{iT}{n} \), \( i \in \{0, \cdots, n\} \) where \( n \geq 1 \). We assume that \( k^n_{i-1}^u = 1 + \sigma^n_{i-1} \sqrt{\Delta t^n_{i-1}} \) and \( k^n_{i-1}^d = 1 - \sigma^n_{i-1} \sqrt{\Delta t^n_{i-1}} \geq 0 \) where \( t \mapsto \sigma_t \) is a positive Lipschitz-continuous function on \([0, T]\). This model implies that \( \text{ess inf}_{n-1} S^n_{t^n_{j}} = k^n_{j-1}^d t^n_{j} S^n_{t^n_{j-1}} \), where for all \( j \in \{1, \cdots, i\} \),

\[ k^n_{i-1}^{u} = \prod_{r=j}^{i} k^n_{r-1}^{u}, \quad k^n_{i-1}^{d} = \prod_{r=j}^{i} k^n_{r-1}^{d}. \]

Note that the assumptions on the multipliers \( k^n_{i-1}^{u} \) and \( k^n_{i-1}^{d} \) imply that

\[ \left| \frac{S^n_{t^n_{i+1}} - 1}{S^n_{t^n_{i}}} \right| \leq \sigma^n_{i} \sqrt{\Delta t^n_{i+1}}, \text{a.s.} \quad (5.17) \]

By Theorem 5.1, we deduce that the infimum super-hedging cost of the European Call option \((S_T - K)^+\) is given by \( h^n(t^n_i, S^n_i) \) defined by (5.14) with terminal condition \( h^n(T, x) = g(x) = (x - K)^+ \). We extend the function \( h^n \) on \([0, T]\) in such a way that \( h^n \) is constant on each interval \([t^n_i, t^n_{i+1}[, \quad i \in \{0, \cdots, n\}\). Such a scheme is proposed by Milstein [24] where a convergence theorem is proved when the terminal condition, i.e. the payoff function is smooth. Precisely, the sequence of functions \((h^n(t, x))_n\) converges uniformly to \( h(t, x) \), solution to the diffusion equation:

\[ \partial_t h(t, x) + \sigma_t^2 \frac{x^2}{2} \partial_{xx} h(t, x) = 0, \quad h(T, x) = g(x). \quad (5.18) \]

In [24], it is supposed that the successive derivatives of the P.D.E.’s solution \( h \) are uniformly bounded. This is not the case for the Call payoff function \( g \). On the contrary the successive derivatives of the P.D.E.’s solution explode at the horizon date, see [23]. In [2], it is proven that the uniform convergence still holds when the payoff function is not smooth provided that the successive derivatives of the P.D.E.’s solution do not explode too much.

Supposing that \( \Delta t^n_i \) is closed to 0, we can identify the observed prices of the Call option with the limit theoretical prices \( h(t, S_t) \) at any instant \( t \), given
by (5.18), to deduce an evaluation of the deterministic function \( t \mapsto \sigma_t \) and test (5.17) on real data. The data set is composed of historical values of the french index CAC 40 and European call option prices of maturity 3 months from the 23rd of October 2017 to the 19th of January 2018. The observed values of \( S \) are distributed as in Figure 1.

For several strikes, matching the observed prices to the theoretical ones derived from the Black and Scholes formula with time-dependent volatility (see (5.18)), we deduce the associated implied volatility \( t \mapsto \sigma_t \) and we compute the proportion of observations satisfying (5.17):

**Fig 1.** Distribution of the observed prices.

**Fig 2.** Ratio of observations satisfying (5.17) as a function of the strike.
When the strike increases less prices’s data are available for the Call option as the strike is too large with respect to the current price $S$, see Figure (1). This could explains the degradation of our results.

5.2.2. Super-hedging prices

We test the infimum super-hedging cost deduced from Theorem 5.1 on some data set composed of historical daily closing values of the french index CAC 40 from the 5th of January 2015 to the 12th of March 2018. The interval $[0,T]$ we choose corresponds to one week composed of 5 days so that the discrete dates are $t^4_i$, $i = 0, \cdots, 4$, i.e. $n = 4$. We first evaluate $\sigma^i$, $i \in \{0, \cdots, 3\}$ as

$$\sigma^i = \max \left( \frac{\left| S^i_{t+1} - S^i_t \right|}{\sqrt{\Delta t^p_{t+1}}} \right) \quad i \in \{0, \cdots, 3\},$$

(5.19)

where $\max$ is the empirical maximum taken over a one year sliding sample window of 52 weeks. Notice that this estimation is model free and does not depend on the strike as it was the case in the preceding sub-section. So we estimate the volatility on 52 weeks, then we implement our hedging strategy on the fifty third one. We then repeat the procedure by sliding the window of one week, i.e. on each of the 112 weeks from the 11th of January 2016 to the 5th of March 2018. We observe the empirical average of the stock price $S_0$ is equal to 4044.

For a payoff function $g(x) = (x - K)^+$, we implement the strategy associated to the super-hedging cost given by Theorem 5.1. The super-hedging cost is given by $h(0,S_0)$ and, using (5.16), we compute the super-hedging strategies $(\theta^*_u)^i_{i \in \{0, \cdots, 3\}}$. We denote by $V_T$ the terminal value of our strategy starting from the minimal price $V_0 = \pi_{0,T} = h(0,S_0)$:

$$V_T = V_0 + \sum_{u=0}^{3} \theta^*_u \Delta S_{t+1}^i.$$ 

We study below the super-hedging error $\varepsilon_T = V_T - (S_T - K)^+$ for different strikes.

Case where $K = 4700$. The distribution of the super-hedging error $\varepsilon_T$ for $K = 4700$ is represented in Figure 3:
The empirical average of the error $\varepsilon_T$ is 12.63 and its standard deviation is 21.65. This result is rather satisfactory in comparison to the large value of the empirical mean of $S_0$ which is equal to 4044. Notice that we observe $E(S_T - K)^+ \approx 282.69$. This empirically confirms the efficiency of our suggested method. The empirical probability of $\{\varepsilon_T < 0\}$ is equal to 15.18% but the Value at Risk 95% is $-10.33$ which confirms that our strategy is conservative.

The empirical average of $V_0/S_0$ is 5.63% and its standard deviation is 5.14%. This is again satisfactory since the theoretical super-hedging price in an incomplete market is often equal to $S_0$ (this is for example the case when
$k^d = 0$ and $k^u = \infty$, in particular when the dynamics of $S$ is modeled by a (discrete) geometric Brownian motion, see [8]).

**Case where $K = S_0$.** We know present the “at the money” case. If we follow the same method given by (5.19), then the empirical average of the error $\varepsilon_T = V_T - (S_T - K)^+$ is 8.1 and its standard deviation is 30.78. We observe $E(S_0) = 4044$, $E(S_T - K)^+ \approx 38.15$, the probability $P(\varepsilon_T < 0) = 8.93\%$ and the Value at Risk 95 % is $-11.41$. The empirical average of $V_0/S_0$ is 2.51% and its standard deviation is 0.53%.

Let us now refine the method. We estimate $k^d_{t^n_i-t^n_{i-1}}$ and $k^u_{t^n_i-t^n_{i-1}}$ as

$$
k^d_{t^n_i-t^n_{i-1}} = \min \frac{S^n_{t^n_i}}{S^n_{t^n_{i-1}}}, \quad k^u_{t^n_i-t^n_{i-1}} = \max \frac{S^n_{t^n_i}}{S^n_{t^n_{i-1}}},
$$

where the empirical minimum and maximum are taken over a one year sliding sample window of 52 weeks, as previously.

**Fig 5. Distribution of the super-hedging error $\varepsilon_T$.**

The empirical average of the error $\varepsilon_T = V_T - (S_T - K)^+$ is 32.8 and its standard deviation is 32.91. This is clearly better than what we get with the first method. The probability $P(\varepsilon_T < 0) = 14.29\%$ and is $P(\varepsilon_T < 0) = 8.93\%$ with the first method. The Value at Risk 95 % is $-10.75$ and it is $-11.41$ with the first method. Let us now focus on the huge loss of 170. This one is observed during the week of the so-called black friday the 24th of June 2016. Large falls of risky assets were observed in European markets mainly
explained by the Brexit vote. In particular, the CAC 40 fell from $S_0 = 4340$ to $S_T = 4106$, with a loss of $-8\%$ on Friday.

The empirical average of $V_0/S_0$ is $1.46\%$ and its standard deviation is $0.5\%$. This means that we have reduced the Call option price whilst improving the hedging error.

6. Appendix

6.1. Conditional support of a vector-valued random variable

We consider a random variable $X$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$, $d \geq 1$, endowed with the Borel $\sigma$-algebra. The goal of this section is to define the conditional support of $X$ with respect to a sub $\sigma$-algebra $\mathcal{H} \subseteq \mathcal{F}$.

**Definition 6.1.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{H}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Let $\mu$ be a $\mathcal{H}$-stochastic kernel (i.e. for all $\omega \in \Omega$, $\mu(\cdot, \omega)$ is a probability on $\mathcal{B}(\mathbb{R}^d)$ and $\mu(A, \cdot)$ is $\mathcal{H}$-measurable, for all $A \in \mathcal{B}(\mathbb{R}^d)$). We define the random set $D_\mu : \Omega \rightarrow \mathbb{R}^d$ by:

$$D_\mu(\omega) := \bigcap \{ A \subset \mathbb{R}^d, \text{ closed, } \mu(A, \omega) = 1 \}.$$  

For $\omega \in \Omega$, $D_\mu(\omega) \subset \mathbb{R}^d$ is called the support of $\mu(\cdot, \omega)$. Let $X \in L^0(\mathbb{R}^d, \mathcal{F})$, we denote by $\text{supp}_\mathcal{H} X$ the set defined in (6.20) when $\mu(A, \omega) = P(X \in A|\mathcal{H})(\omega)$ is the regular version of the conditional law of $X$ knowing $\mathcal{H}$ and we call it the conditional support of $X$ with respect to $\mathcal{H}$. 

![Fig 6. Distribution of the ratio $V_0/S_0$.](image)
Remark 6.2. This notion reduces to the usual definition of the support of $X$ when $\mathcal{H}$ is the trivial sigma-algebra (see p441 of [1]). Using Theorems 12.7 and 12.14 of [1], we have that $\mu(\cdot, \omega)$ admits a unique support $D_\mu(\omega) \subset \mathbb{R}^d$ and that $\mu(D_\mu(\omega), \omega) = 1$.

Lemma 6.3. $D_\mu$ is non-empty, closed-valued, $\mathcal{H}$-measurable and graph-measurable random set (i.e. $\text{Graph}(D_\mu) \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$).

Proof. It is clear from the definition (6.20) that for all $\omega \in \Omega$, $D_\mu(\omega)$ is a non-empty and closed subset of $\mathbb{R}^d$. We now show that $D_\mu$ is $\mathcal{H}$-measurable. Let $O$ be a fixed open set in $\mathbb{R}^d$ and $\mu_O : \omega \in \Omega \mapsto \mu_O(\omega) := \mu(O, \omega)$. As $\mu$ is a stochastic kernel, $\mu_O$ is $\mathcal{H}$-measurable. By definition of $D_\mu(\omega)$ we get that $\{\omega \in \Omega, D_\mu(\omega) \cap O \neq \emptyset\} = \{\omega \in \Omega, \mu_O(\omega) > 0\} \in \mathcal{H}$, and $D_\mu$ is $\mathcal{H}$-measurable. Now using Theorem 14.8 of [26], $\text{Graph}(D_\mu) \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$ (recall that $D_\mu$ is closed-valued) and $D_\mu$ is $\mathcal{H}$-graph-measurable. $\square$

6.2. Conditional essential supremum

A very general concept of conditional essential supremum of a family of vector-valued random variables is defined in [20, Definition 3.1 and Lemma 3.9] with respect to a random partial order. In the real case, a generalization of the definition of essential supremum (see [19, Section 5.3.1] for the definition and the proof of existence of the classical essential supremum) is given by the following result:

Proposition 6.4. Let $\mathcal{H} \subseteq \mathcal{F}$ be two $\sigma$-algebras on a probability space. Let $\Gamma = (\gamma_i)_{i \in I}$ be a family of real-valued $\mathcal{F}$-measurable random variables. There exists a unique $\mathcal{H}$-measurable random variable $\gamma_\mathcal{H} \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$ denoted $\text{ess sup}_{\mathcal{H}} \Gamma$ which satisfies the following properties:

1. For every $i \in I$, $\gamma_\mathcal{H} \geq \gamma_i$ a.s.
2. If $\zeta \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$ satisfies $\zeta \geq \gamma_i$ a.s. $\forall i \in I$, then $\zeta \geq \gamma_\mathcal{H}$ a.s.

One can also consult [3] where the conditional supremum is defined in the case where $I$ is a singleton.

Proof. The proof is given for sake of completeness and pedagogical purpose. The authors thanks T. Jeulin who suggested this (elegant) proof. Considering the homeomorphism $\arctan$ we can restrict our-self to $\gamma_i$ taking values in $[0, 1]$. We denote by $P_{\gamma_i|\mathcal{H}}$ a regular version of the conditional law of $\gamma_i$ knowing $\mathcal{H}$. Let $\zeta \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$ such that $\zeta \geq \gamma_i$ a.s. $\forall i \in I$. It is easy to see
that $\zeta \geq \gamma_i$ a.s. $\iff P_{\zeta|\mathcal{H}}([\kappa, \infty, x])|_{x=\zeta} = 1$ a.s. and $\text{supp}_{\mathcal{H}} \gamma_i \subset ]-\infty, \zeta]$ a.s. follows from Definition 6.1. Let $\Lambda_{\gamma_i|\mathcal{H}} = \sup\{x \in [0, 1], x \in \text{supp}_{\mathcal{H}} \gamma_i\}$ then $\Lambda_{\gamma_i|\mathcal{H}} \leq \zeta$ a.s. It is easy to see that $\Lambda_{\gamma_i|\mathcal{H}}$ is $\mathcal{H}$-measurable. So taking the classical essential supremum, we get that $\text{ess sup}_{\gamma_i} \Lambda_{\gamma_i|\mathcal{H}} \leq \zeta$ a.s. and that $\text{ess sup}_{\gamma_i} \Lambda_{\gamma_i|\mathcal{H}}$ is $\mathcal{H}$-measurable. We conclude that $\gamma_{\mathcal{H}} = \text{ess sup}_{\gamma_i} \Lambda_{\gamma_i|\mathcal{H}}$ a.s. since for every $i \in I$, $P(\gamma_i \in \text{supp}_{\mathcal{H}} \gamma_i) = 1$.

**Remark 6.5.** Let $Q$ be an absolutely continuous probability measure with respect to $P$. Let $Z = \frac{dQ}{dP}$ and $E_Q$ be the expectation under $Q$. As for every $i \in I$, $\text{ess sup}_{\mathcal{H}} \Gamma \geq \gamma_i$ a.s. $\text{ess sup}_{\mathcal{H}} \Gamma$ is $\mathcal{H}$-measurable,

$$\text{ess sup}_{\mathcal{H}} \Gamma \geq \frac{E(Z\gamma_i|\mathcal{H})}{E(Z|\mathcal{H})} = E_Q(\gamma_i|\mathcal{H}).$$

Inspired by Theorem 2.8 in [3], we may easily show the following tower property:

**Lemma 6.6.** Let $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{F}$ be $\sigma$-algebras on a probability space and let $\Gamma = (\gamma_i)_{i \in I}$ be a family of real-valued $\mathcal{F}$-measurable random variables. Then,

$$\text{ess sup}_{\mathcal{H}_1} \left( \text{ess sup}_{\mathcal{H}_2} \Gamma \right) = \text{ess sup}_{\mathcal{H}_1} \Gamma.$$

### 6.3. Link between two notions

Our goal is to extend the the fact that (see the proof of Proposition 6.4)

$$\text{ess sup}_{\mathcal{H}} X = \sup_{x \in \text{supp}_{\mathcal{H}} X} x$$

a.s.

First we show two useful lemmata.

**Lemma 6.7.** Let $\mathcal{K} : \Omega \rightarrow \mathbb{R}^d$ be a $\mathcal{H}$-measurable and closed-valued random set such that $\text{dom} \mathcal{K} = \Omega$ and let $h : \Omega \times \mathbb{R}^k \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function such that $h(\omega, x, \cdot)$ is l.s.c. for all $(\omega, x) \in \Omega \times \mathbb{R}^k$. Then $(\omega, x) \in \Omega \times \mathbb{R}^k \rightarrow s(\omega, x) = \sup_{z \in \mathcal{K}(\omega)} h(\omega, x, z)$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$-measurable.

**Proof.** Let us consider a Castaing representation $(\eta_n)_{n \in \mathbb{N}}$ of $\mathcal{K}$, i.e. we have $\mathcal{K}(\omega) = \text{cl}\{\eta_n(\omega), n \in \mathbb{N}\}$ where the closure is taken in $\mathbb{R}^d$ and $\eta_n(\omega) \in \mathcal{K}(\omega)$ for all $n$ and $\omega$ (the $\eta_n$ are defined on the whole space $\Omega$ since $\text{dom} \mathcal{K} = \Omega$).

Fix some $c \in \mathbb{R}$, we get that

$$\{ (\omega, x) \in \Omega \times \mathbb{R}^d, s(\omega, x) \leq c \} = \bigcap_n \{ (\omega, x) \in \Omega \times \mathbb{R}^d, h(\omega, x, \eta_n(\omega)) \leq c \}.$$
Indeed the first inclusion follows from the fact that $\eta_n(\omega) \in \mathcal{K}(\omega)$ for all $n$ and all $\omega$. For the reverse inclusion, fix some $(\omega, x) \in \bigcap_n \{(\omega, x), h(\omega, x, \eta_n(\omega)) \leq c\}$. For any $z \in \mathcal{K}(\omega)$ one gets that $z = \lim_n \eta_n(\omega)$. Then, we get that $h(\omega, x, z) \leq \liminf h(\omega, x, \eta_n(\omega)) \leq c$ Now recalling that $h$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and that $\eta_n$ is $\mathcal{H}$-measurable, we deduce that $(\omega, x) \mapsto h(\omega, x, \eta_n(\omega))$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$-measurable and $s$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$-measurable. $\square$

**Lemma 6.8.** Let $\mathcal{K} : \Omega \to \mathbb{R}^d$ be a $\mathcal{H}$-measurable and closed-valued random set such that $\text{dom } \mathcal{K} = \Omega$ and let $h : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be l.s.c. in $x$. Then,

$$\sup_{x \in \mathcal{K}} h(x) = \sup_n h(\eta_n), \quad (6.21)$$

where $(\eta_n)_n$ is a Castaing representation of $\mathcal{K}$.

**Proof.** Let $\omega \in \Omega$. As $(\eta_n(\omega))_n \subset \mathcal{K}(\omega)$, $h(\omega, \eta_n(\omega)) \leq \sup_{x \in \mathcal{K}(\omega)} h(\omega, x)$ and thus $\sup_n h(\eta_n) \leq \sup_{x \in \mathcal{K}} h(x)$. Let $x \in \mathcal{K}(\omega) = \text{cl}\{\eta_n(\omega), n \in \mathbb{N}\}$, by lower semicontinuity of $h$, $h(\omega, x) \leq \liminf h(\omega, \eta_n(\omega)) \leq \sup_n h(\omega, \eta_n(\omega))$. We conclude that $\sup_{x \in \mathcal{K}} h(x) \leq \sup_n h(\eta_n)$ and (6.21) is proved. $\square$

**Lemma 6.9.** Let $X \in L^0(\mathbb{R}^d, \mathcal{F})$ such that $\text{dom } \text{supp}_\mathcal{H}X = \Omega$ and let $h : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function which is l.s.c. in $x$. Then,

$$\text{ess sup}_\mathcal{H} h(X) = \sup_{x \in \text{supp}_\mathcal{H}X} h(x) = \sup_n h(\gamma_n) \text{ a.s.}, \quad (6.22)$$

where $(\gamma_n)_{n \in \mathbb{N}}$ is a Castaing representation of $\text{supp}_\mathcal{H}X$.

**Proof.** As $P(X \in \text{supp}_\mathcal{H}X|\mathcal{H}) = 1$ we have that $\sup_{x \in \text{supp}_\mathcal{H}X} h(x) \geq h(X)$ a.s. and by definition of $\text{ess sup}_\mathcal{H} h(X)$, we get that $\sup_{x \in \text{supp}_\mathcal{H}X} h(x) \geq \text{ess sup}_\mathcal{H} h(X)$ a.s. since $\sup_{x \in \text{supp}_\mathcal{H}X} h(x)$ is $\mathcal{H}$-measurable by Lemmata 6.3 and 6.7.

Let $(\gamma_n)_n$ be a Castaing representation of $\text{supp}_\mathcal{H}X$, Lemma 6.8 implies that $\sup_{x \in \text{supp}_\mathcal{H}X} h(x) = \sup_n h(\gamma_n)$. Fix some rational number $\varepsilon > 0$ and set $Z_\varepsilon = 1_{B(\gamma_n, \varepsilon)}(X)$, where $B(\gamma_n, \varepsilon)$ is the closed ball of center $\gamma_n$ and radius $\varepsilon$. Note that $E(Z_\varepsilon|\mathcal{H}) = P(X \in B(\gamma_n, \varepsilon)|\mathcal{H}) > 0$. Indeed if it does not hold true $P(X \in \mathbb{R}^d \setminus B(\gamma_n, \varepsilon)|\mathcal{H}) = 1$ on some $H \in \mathcal{H}$ such that $P(H) > 0$ and by definition 6.1, $\text{supp}_\mathcal{H}X \subset \mathbb{R}^d \setminus B(\gamma_n, \varepsilon)$ on $H$, which contradicts $\gamma_n \in \text{supp}_\mathcal{H}X$. By definition of the essential supremum again, we have that $\text{ess sup}_\mathcal{H} h(X) \geq h(X)$ a.s. and that $\text{ess sup}_\mathcal{H} h(X)$ is $\mathcal{H}$-measurable. So we
obtain for all fixed $\omega \in \Omega_\varepsilon$ where $\Omega_\varepsilon$ is of full measure that
\[
\text{ess sup}_H h(X)(\omega) \geq \frac{\mathbb{E}(Z_\varepsilon h(X)|\mathcal{H})}{\mathbb{E}(Z_\varepsilon|\mathcal{H})}(\omega) = \frac{\int \mathbb{1}_{B(\gamma_n(\omega),\varepsilon)}(x) h(\omega, x) P_{X|\mathcal{H}}(dx; \omega)}{\mathbb{E}(Z_\varepsilon|\mathcal{H})}(\omega)
\]
\[
\geq \frac{\int (\inf_{y \in B(\gamma_n(\omega),\varepsilon)} h(\omega, y)) \mathbb{1}_{B(\gamma_n(\omega),\varepsilon)}(x) P_{X|\mathcal{H}}(dx; \omega)}{\mathbb{E}(Z_\varepsilon|\mathcal{H})}(\omega)
\]
\[
\geq \inf_{y \in B(\gamma_n(\omega),\varepsilon)} h(\omega, y).
\]
So on the full measure set $\bigcap_{n \in \mathbb{Q}, \varepsilon > 0} \Omega_\varepsilon$, using that $h$ is l.s.c. (recall [26, Definition 1.5, equation 1(2)]), we have that
\[
\lim_{\varepsilon \to 0} \inf_{x \in B(\gamma_n,\varepsilon)} h(x) = \liminf_{x \to \gamma_n} h(x) = h(\gamma_n)
\]
and it follows that $\text{ess sup}_H h(X) \geq h(\gamma_n)$ a.s. Taking the supremum over all $n$, we get that $\text{ess sup}_H h(X) \geq \sup_n h(\gamma_n) = \sup_{x \in \text{supp}_H X} h(x)$ a.s. □

We have the following easy extension:

**Lemma 6.10.** Let $\mathcal{X} \subset L^0(\mathbb{R}^d, \mathcal{F})$ such that $\text{dom supp}_H X = \Omega$ for all $X \in \mathcal{X}$ and $\cup_{X \in \mathcal{X}} \text{supp}_H X$ is a $\mathcal{H}$-measurable and closed-valued random set. Let $h : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function which is l.s.c. in $x$. Then,
\[
\text{ess sup}_H \{h(X), X \in \mathcal{X}\} = \sup_{x \in \cup_{X \in \mathcal{X}} \text{supp}_H X} h(x) \text{ a.s.} \quad (6.23)
\]

Note that if $\mathcal{X}$ is countable, $\cup_{X \in \mathcal{X}} \text{supp}_H X$ is clearly $\mathcal{H}$-measurable. If $\mathcal{X} = L^0(\mathbb{R}^d, \mathcal{F})$, then $\cup_{X \in \mathcal{X}} \text{supp}_H X = \mathbb{R}^d$, which is again $\mathcal{H}$-measurable and also closed-valued.

**Proof:** For all $X \in \mathcal{X}$, $\text{ess sup}_H \{h(X), X \in \mathcal{X}\} \geq h(X)$ a.s. and as $\text{ess sup}_H \{h(X), X \in \mathcal{X}\}$ is $\mathcal{H}$-measurable, we get that $\text{ess sup}_H \{h(X), X \in \mathcal{X}\}$ a.s. and also $\text{ess sup}_H \{h(X), X \in \mathcal{X}\} \geq \sup_{X \in \mathcal{X}} \text{ess sup}_H h(X)$ a.s. Conversely, for all $X \in \mathcal{X}$, $\text{sup}_{X \in \mathcal{X}} \text{ess sup}_H h(X) \geq h(X)$ a.s. and if $\text{sup}_{X \in \mathcal{X}} \text{ess sup}_H h(X)$ is $\mathcal{H}$-measurable, we conclude that $\text{sup}_{X \in \mathcal{X}} \text{ess sup}_H h(X) \geq \text{ess sup}_H \{h(X), X \in \mathcal{X}\}$ a.s. Using Lemma 6.9, we get that
\[
\sup_{X \in \mathcal{X}} \text{ess sup}_H h(X) = \sup_{X \in \mathcal{X}} \sup_{x \in \text{supp}_H X} h(x) = \sup_{x \in \cup_{X \in \mathcal{X}} \text{supp}_H X} h(x) \text{ a.s.}
\]
Since $\cup_{X \in \mathcal{X}} \text{supp}_H X$ is $\mathcal{H}$-measurable and closed-valued, Lemma 6.7 implies that $\sup_{x \in \cup_{X \in \mathcal{X}} \text{supp}_H X} h(x)$ is $\mathcal{H}$-measurable and the proof is complete. □
Lemma 6.11. Consider $X \in L^0(\mathbb{R}_+, \mathcal{F})$. Then, we have a.s. that

\[ \text{ess inf}_{\mathcal{H}} X = \inf \text{supp}_{\mathcal{H}} X, \quad \text{ess sup}_{\mathcal{H}} X = \sup \text{supp}_{\mathcal{H}} X, \]

\[ \text{ess inf}_{\mathcal{H}} X \in \text{supp}_{\mathcal{H}} X, \quad \text{on the set } \{ \text{ess inf}_{\mathcal{H}} X > -\infty \}, \]

\[ \text{ess sup}_{\mathcal{H}} X \in \text{supp}_{\mathcal{H}} X, \quad \text{on the set } \{ \text{ess sup}_{\mathcal{H}} X < \infty \}. \]

Proof. The two first statements are deduced from the construction of $\text{ess sup}_{\mathcal{H}} X$ in Proposition 6.4. Suppose that $\text{ess inf}_{\mathcal{H}} X \notin \text{supp}_{\mathcal{H}} X$ on some non-null measure subset $\Lambda \in \mathcal{H}$ of $\{ \text{ess inf}_{\mathcal{H}} X > -\infty \}$. As $\text{supp}_{\mathcal{H}} X$ is $\mathcal{H}$-measurable and closed-valued, by a measurable selection argument, we deduce the existence of $r \in L^0(\mathbb{R}_+, \mathcal{H})$ such that $r > 0$ and $(\text{ess inf}_{\mathcal{H}} X - r, \text{ess inf}_{\mathcal{H}} X + r) \subseteq \mathbb{R}\setminus\text{supp}_{\mathcal{H}} X$ on $\Lambda$. As $X \in \text{supp}_{\mathcal{H}} X$ a.s. and $X \geq \text{ess inf}_{\mathcal{H}} X$ a.s., we deduce that $X \geq \text{ess inf}_{\mathcal{H}} X + r$ on $\Lambda$, which contradicts the definition of $\text{ess inf}_{\mathcal{H}} X$. The last statement is similarly shown. \(\Box\)

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