Diffusion equations: convergence of the functional scheme derived from the binomial tree with local volatility for non smooth payoff functions.

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Abstract: The function solution to the functional scheme derived from the binomial tree financial model with local volatility converges to the solution of a diffusion equation of type $h_t(t, x) + \frac{x^2 \sigma^2(t, x)}{2} h_{xx}(t, x) = 0$ as the number of discrete dates $n \to \infty$. Contrarily to classical numerical methods, in particular finite difference methods, the principle behind the functional scheme is only based on a discretization in time. We establish the uniform convergence in time of the scheme and provide the rate of convergence when the payoff function is not necessarily smooth as in finance. We illustrate the convergence result and compare its performance to the finite difference and finite element methods by numerical examples.

Keywords and phrases: Binomial tree model, European option pricing, Diffusion partial differential equations, Finite difference scheme, Finite element scheme.

1. Introduction

Our present contribution originates from the binomial financial market model with $T$ steps defined as follows. Let $(\Omega, (\mathcal{F}_t)_{t=0,\ldots,T}, P)$ be a complete stochastic basis. We consider an adapted price process $(S_t)_{t=0,\ldots,T}$ where $S_0$ is given at time 0. Moreover, we suppose that $P(S_{t+1} = S_t k^u_t(S_t) | \mathcal{F}_t) = P(S_{t+1} = S_t k^d_t(S_t) | \mathcal{F}_t) = 1/2$ where $0 \leq k^d_t < 1 < k^u_t$ are two functions such that
\( k^d + k^u = 2 \). Under these assumptions, we deduce that \((S_t)_{t=0,\ldots,T}\) is a \( P \)-martingale, i.e. the financial market model satisfies the classical no-arbitrage condition characterized in the Dalang–Morton-Willinger theorem [11].

A discrete-time portfolio process \( V \) is by definition a process adapted to the filtration \((\mathcal{F}_t)_{t=0,\ldots,T}\) which satisfies the dynamics
\[
\Delta V_t := V_t - V_{t-1} = \theta_{t-1} \Delta S_t, \quad t = 1, \ldots, T,
\]
where \( \theta_t \) is \( \mathcal{F}_t \)-measurable for all \( t = 0, \ldots, T-1 \). Moreover, it replicates a contingent claim \( g(S_T) \) if \( V_T = g(S_T) \). When \( g(S_T) \) is integrable, we easily deduce that \( V \) is a martingale and, by induction, we get that \( V_t = C(t, S_t) \) for some measurable function \( C \). Using the martingale property \( V_t - V_{t-1} = \mathbb{E}(V_t|\mathcal{F}_t) \), we finally deduce that
\[
C(t-1, S_{t-1}) = \frac{1}{2} C(t, S_{t-1} k^d_{t-1}) + \frac{1}{2} C(t, S_{t-1} k^u_{t-1}), \quad t = 1, \ldots, T. \tag{1.1}
\]

Actually, in the recent paper [4], a more general model is considered. At time \( t+1 \), it is only supposed that the price \( S_{t+1} \) may take any value between \( S_t k^d(S_t) \) and \( S_t k^u(S_t) \). It is shown that the relation (1.1) also defines the (minimal) super-replicating portfolio process replicating \( g(S_T) \).

The natural idea is to study this scheme, as the number \( n + 1 \) of dates \( (Ti/n)_{i=0,\ldots,n} \) tends to \( \infty \), when we discretize the continuous-time interval \([0, T]\). In the next section, we consider the case where \( k^d_t(x) = 1 - \sigma(t, x) \sqrt{T/n} \) and \( k^u_t(x) = 1 + \sigma(t, x) \sqrt{T/n} \). Actually, the asymptotic behaviour of such binomial model is well known, at least for the price process. Indeed, let us consider the continuous-time price process \( S^n \) of the binomial model defined by \( S^n_t = S^n_{T(n-1)} \) if \( t \in [T(n-1), T(n)] \) where \( S^n_{T(n-1)} \) is defined by (1.1) with \( k^d \) and \( k^u \) and the initial value \( S_0 \) at \( t = 0 \). Then, by [22, Proposition 3.2.1] under mild conditions on \( \sigma \), the sequence \( S^n \) weakly converges to the diffusion process \( S \) satisfying the stochastic differential equation
\[
dS_t = S_t \sigma(t, S_t) dW_t, \tag{1.2}
\]
where \( W \) is a standard Brownian motion. We also deduce the convergence of the price functions \( (C^n)_{n \geq 1} \) given by (1.1) towards the limit price \( C \) of the continuous-time model given by (1.2). Recall that \( C \) is the solution to the diffusion equation (1.3). This has been proved when the underlying asset follows a geometric Brownian motion in the papers [17], [16] and [10] in the case of the Call option. When \( \sigma \) is constant, the convergence result is extended to a large class of payoff functions \( g \) in [3, Chapter 4] where it is also proved that the best rate we can get in general is \( \frac{1}{\sqrt{n}} \), see also [27] and
the papers [7], [14] and [26]. The paper [20] considers local volatility functions with smooth terminal conditions while a recombining binomial tree scheme is introduced in [21] to obtain the convergence for sufficiently smooth price functions.

Our main contribution is to consider only local diffusion coefficients and non-smooth payoff functions. To do so, we adopt a condition (Condition D) under which the solution $C$ of the diffusion equation may admit unbounded derivatives. We show the convergence of the scheme with rate $n^{-1/2}$. Moreover, numerical experiments are presented to confirm the accuracy of the functional scheme. In particular, we compare them to the (FD) and (FE) methods. We present some examples where the functional scheme appears to outperform the finite difference method and, moreover, it is well adapted to parallel computing.

Actually, there are three major methods to numerically approximate the solution of a diffusion equation of type

$$h_t(t, x) + \frac{x^2 \sigma^2(t, x)}{2} h_{xx}(t, x) = 0, \quad t \in [0, T), \quad (1.3)$$

with the boundary condition $h(T, x) = g(x)$. The first one is to use the Monte Carlo methods as the solution $h$ admits a probabilistic representation, see [13]. The second one is to numerically compute the solution directly from the PDE. In particular, the very well known finite difference (FD) technique is based on approximations of the successive derivatives. It requires a discretization of some compact subset $[0, T] \times [a, b], a \leq b$, both in time and in the space variable. As we need to fix extra boundary conditions, when $x = a$ or $x = b$, a second type of approximation error may appear. For instance, if $b$ is large enough, we generally set the condition $h(t, b) = \lim_{x \to \infty} h(t, x)$ while the condition $h(t, 0) = g(0)$ is chosen if $a = 0$ under some mild conditions on $\sigma$. There are a lot of articles in the literature focusing on this technique, see for example [3] and [25]. The finite element (FE) method [6] and other techniques as finite volume [12] or spectral methods [23], [2], are more sophisticated but they are also based on a discretization of the space variable.

The third one is based on binomial trees. Such a scheme is proposed by Milstein [20] where a convergence theorem is proved when the terminal condition, i.e. the payoff function in finance, is smooth. In particular, it is supposed that the successive derivatives of the solution $h$ of the P.D.E. are uniformly...
bounded. In finance, e.g. for the Call payoff function $g(x) = (x - K)^+$, this is not the case. On the contrary the successive derivatives of the P.D.E. solution explode at the horizon date [18]. Note that this third method is only based on a discretization in time of the interval $[0, T]$ and allows to simulate $t \mapsto h(t, x)$ for a fixed $x$. The advantage is an ease of implementation as the functional scheme is basically defined. Precisely, the numerical scheme we study is defined by two functions depending on the number $n$ of discretization dates. We focus here on a particular choice of such functions which lead to a uniform approximation in time of the diffusion equation under some mild conditions. An open problem is to study the suggested functional scheme more generally.

2. Functional scheme for diffusion equations

Let us consider a bounded diffusion function $\sigma : [0, T] \times \mathbb{R}_+$ and the associated backward parabolic equation

$$h_t(t, x) + \frac{x^2 \sigma^2(t, x)}{2} h_{xx}(t, x) = 0, \quad t \in [0, T), \quad h(T, x) = g(x), \quad (2.4)$$

where the terminal condition is defined by a Lipschitz function $g$ with Lipschitz constant $L_g$. Note that $h_t$ and $h_{xx}$ are respectively the first and the second derivatives of $h$ with respect to time $t$ and space variable $x \in \mathbb{R}$. This equation is very well known in physics but also in mathematical finance for models without friction when the risky asset $S$ is driven by a standard Brownian motion $W$ so that it satisfies the stochastic differential equation $dS_t = \sigma(t, S_t) S_t dW_t$ under a risk neutral probability measure $P$ \footnote{If we suppose that the risk-free interest rate is $r = 0$}. In that case, $h(t, S_t)$ is the value at time $t$ of the unique self-financing portfolio process $V$ composed of a fraction of the risk-less bond $B = 1$ and the risky asset $S$ such that it replicates the European option payoff $g(S_T)$, i.e. $V_T = g(S_T)$. The quantity $\sigma(t, S_t)$ is then interpreted as a local volatility coefficient.

In the following, we consider the uniform grid on $[0, T]$ given by $t_i^n = (T/n)i$, $i = 0, \cdots, n$ where $n \geq 1$. We then define the following functions:
\[ k^{n+}_t(x) := 1 + \sigma(t, x)\sqrt{\frac{T}{n}}, \quad k^{n-}_t(x) := 1 - \sigma(t, x)\sqrt{\frac{T}{n}}, \quad (2.5) \]

\[ \lambda^n_t(x) := \frac{1 - k^{n-}_t(x)}{k^{n+}_t(x) - k^{n-}_t(x)} = \frac{1}{2}, \quad \mu^n_t(x) := 1 - \lambda^n_t(x) = \frac{1}{2}. \]

Let \( h^n \) be the piecewise constant function defined on \([0, T]\) by \( h^n(t, x) = h^n(t_{i-1}, x) \) if \( t \in [t^n_{i-1}, t^n_i], i \leq n \), where the sequence \((h^n(t_i, x))_{i \leq n}\) is defined recursively by the following functional scheme:

\[
\begin{align*}
    h^n(t_{i-1}, x) &= \lambda^n_{t_{i-1}}(x) h^n(t_i, k_i^{n-}(x)) + \mu^n_{t_{i-1}}(x) h^n(t_i, k_i^{n+}(x)), \\
    h^n(T, x) &= g(x), \quad i \leq n. 
\end{align*}
(2.6)
\]

Our main result states that the sequence of functions \((h^n)_{n \in \mathbb{N}}\) uniformly converges to \( h \) as \( n \to \infty \) under some mild conditions (called Condition D below) satisfied by the successive derivatives of \( h \), solution to (2.4). Of course, the functional scheme (2.6) may also be considered for other choices of functions \( k^{n-} \) and \( k^{n+} \). In this paper, we restrict ourselves to the functions defined by (2.5), i.e. when the coefficients defining the binomial model are symmetric, but it is an interesting open problem to study such a scheme for more general coefficients \( k^{n-} \) and \( k^{n+} \).

Recall that \( h(t, x) = \mathbb{E}g(S_{x,t}(T)) \) where, for \( t \leq T \), \( S_{x,t} \) is the unique solution to the stochastic differential equation (5.17) on the interval \([t, T]\) with initial condition \( S_{x,t}(t) = x \). We deduce that \( h(t, 0) = g(0) \). In the following, we suppose that \( \sigma \) is locally Lipschitz and bounded so that existence and uniqueness holds for (5.17), see for instance [13, Theorem 2.2, p104].

The conditions we consider below are satisfied by a large class of payoff functions. Indeed, these inequalities have been proved for finance in the paper [18, Theorem 4.1] (see also see also [9, Section 5]) in the case where the diffusion coefficient \( \sigma \) is sufficiently smooth while the payoff function \( g \) is continuous even if \( g \) is not necessarily differentiable while \( g'' \) may not exist. In particular, it is not supposed that the successive derivatives are bounded contrarily to [20].
Condition D:

There exists a constant $C > 0$ such that:

\[
\begin{align*}
|h_x(t, x)| &\leq C, \quad |h_{xx}(t, x)| \leq \frac{C}{x \sqrt{T - t}}, \\
|h_{xxx}(t, x)| &\leq \frac{C}{x^2 (T - t)}, \quad |h_{xt}(t, x)| \leq \frac{C}{T - t}, \\
|h_{xxt}(t, x)| &\leq \frac{C}{|x| (T - t)^{3/2}}, \quad |h_{tt}(t, x)| \leq \frac{C |x|}{(T - t)^{1/2}}, \\
|h_{ttt}(t, x)| &\leq \frac{C |x|^2}{(T - t)^{5/2}}, \quad |h_{xtt}(t, x)| \leq \frac{C}{(T - t)^2}.
\end{align*}
\]

In the following, we obtain a non asymptotic estimation of the deviation between $h^n$ and $h$:

**Theorem 2.1.** Suppose that $\sigma$ is a bounded Lipschitz function and assume that the solution to Equation (2.4) satisfies Condition (D). Then, there exists a constant $C > 0$, depending on $g$ and $\sigma$ and independent of $x$, such that

\[
\sup_{t \in [0, T]} |h^n(t, x) - h(t, x)| \leq \frac{C|x|}{\sqrt{n}}, \quad n > 0.
\]

The proof is deduced from the results proved in Section 4. Note that, by choosing specific discretization dates, see [8], we should improve the convergence rate of the functional scheme. This is left for future research.

3. Numerical examples

In the following, we compare the performance of the functional scheme to the Crank-Nicolson finite difference method and the finite element method on some examples. We start by recalling shortly the implementation of the Crank-Nicolson finite difference method.

3.1. Description of the Cranck-Nicolson scheme and Rannacher time-marching for the finite difference method

For the Black and Scholes model with volatility function $\sigma(S, t)$, we want to numerically approximate the solution to the equation
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S,t) S^2 \frac{\partial^2 V}{\partial S^2} = 0, \quad V(T, S) = g(S).
\]

Through the logarithm transformation of the asset value \(x = \log(S)\) and reversing time \(\tau = T - t\), we deduce that \(u(\tau, x) = V(T - t, e^x)\) is solution to:

\[
\frac{\partial u}{\partial \tau} - \frac{1}{2} \sigma^2(x, \tau) \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \sigma^2(x, \tau) \frac{\partial u}{\partial x} = 0.
\]

In the following examples, we only consider the \(u_0(x) = g(e^x)\). Thus, the solution should satisfy \(u(\tau, x) \approx 0\) as \(x \to -\infty\) and \(u(\tau, x) \approx \exp(x)\) as \(x \to +\infty\). This equation is solved on a domain \([0, T] \times [a, b]\). This is why we actually solve the approximating function solution to

\[
\begin{aligned}
\frac{\partial u}{\partial \tau} - \frac{1}{2} \sigma^2(x, \tau) \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \sigma^2(x, \tau) \frac{\partial u}{\partial x} = 0, \\
u(0, x) = u_0(x), \\
u(\tau, a) = 0, \\
u(\tau, b) = \exp(b),
\end{aligned}
\]

where, for each timestep \(j\), \(A^{(j)}\) is a tridiagonal matrix defined by:

\[
\begin{align*}
A^{(j)}_{n,n} &= \frac{\sigma^2_{j,n}}{h^2} \\
A^{(j)}_{n+1,n} &= -\frac{\sigma^2_{j,n}}{4h} - \frac{\sigma^2_{j,n}}{2h^2} \\
A^{(j)}_{n,n+1} &= \frac{\sigma^2_{j,n}}{4h} - \frac{\sigma^2_{j,n}}{2h^2}
\end{align*}
\]

The vector \(F\) is defined by \(F^j_n = 0\) for \(0 \leq n \leq N\) and

\[
F^{(j)}_{N+1} = k \exp(b) \left( \frac{-\sigma^2_{j,N+1}}{2h} - \frac{\sigma^2_{j,N+1}}{2h^2} \right).
\]
When the payoff function is not smooth, it has been observed [5] that the Crank-Nicolson finite difference method does not perform very well. This is in particular due to the irregularity of the first and second derivatives of the payoff function. The backward Euler damping, also called Rannacher time stepping, is then proposed by Rannacher [24] which recovers the second-order convergence by replacing the Crank-Nicolson method for the first time step by two half-timesteps using Backward Euler time integration. Actually, the use of four half-timesteps appears to be optimal [5]. We follow this recommendation.

3.2. Description of the implicit Euler scheme for the finite element method of degree one

The finite element method described here is largely inspired by the one presented in [1]. We first set up the weak formulation of the problem. Let us introduce the space $V$ defined by

$$V = \left\{ v \in L^2(\mathbb{R}^2) : S \frac{dv}{dS} \in L^2(\mathbb{R}^2) \right\}$$

where the derivative must be understood in the sense of the distributions on $\mathbb{R}_+$. Then $(v, w)_V = (v, w) + \left( S \frac{dv}{dS}, S \frac{dw}{dS} \right)$ is a scalar product on $V$ so that the space $V$ is a Hilbert space. We denote by $V'$ the topological dual space of $V$, and for $w \in V'$, the norm is given by $\| w \|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{(w, v)}{\| v \|_V}$.

Recall that we are interested in the P.D.E.

$$\frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2(S, t) \frac{d^2 u}{dS^2} = 0, \quad u(0, S) = g(S). \quad (3.7)$$

The equation (3.7) is multiplied by a smooth real valued function $w$ defined on $\mathbb{R}^+$ before being integrated with respect to the variable $S$ on $\mathbb{R}^+$. By the integration by part formula, we get that

$$\frac{d}{dt} \left( \int_{\mathbb{R}^+} u(S, t) w(S) dS \right) + a_t (v, w) = 0$$

where the bilinear from $a_t$ is defined by

$$a_t (v, w) = \int_{\mathbb{R}^+} \left( \frac{1}{2} S^2 \sigma^2(S, t) \frac{\partial v}{\partial S} \frac{\partial w}{\partial S} \right) dS$$

$$+ \int_{\mathbb{R}^+} \left( \sigma^2(S, t) + S \frac{\partial \sigma}{\partial S}(S, t) \right) S \frac{\partial v}{\partial S} w dS \quad (3.8)$$
Let us define $C^0([0, T], V)$ the space of continuous $V$-valued functions and $L^2([0, T], V')$ the space of square integrable $V$-valued functions. With $g \in L^2(\mathbb{R}_+)$, it has be proven in [19] that it is possible to write a weak formulation of (3.7):

**Weak formulation** Find $u \in C^0([0, T], V) \cap L^2([0, T], V')$ with $\frac{\partial u}{\partial t} \in L^2([0, T], V')$, and $u|_{t=0} = g$ in $\mathbb{R}_+$, and for a.e. $t \in (0, T)$, for all $v \in V$:

$$\left( \frac{\partial u}{\partial t}(t), v \right) + a_t(u(t), v) = 0. \quad (3.9)$$

Under the assumptions that there exists a positive constant $\sigma_{\text{max}}$ such that $|\sigma(S, t)| \leq \sigma_{\text{max}}$ and a positive constant $C_{\sigma}$ such that $|S \frac{\partial \sigma}{\partial S}(S, t)| \leq C_{\sigma}$ for all $t \in [0, T]$ and all $S \in \mathbb{R}_+$, the weak formulation admits a unique solution. This can be applied to a Put function which is $L^2(\mathbb{R}_+)$. Therefore, we compute the price for a put option and then use the call-put parity to deduce the call price.

For a numerical approximation of $u$, the price domain is restricted to $(0, \overline{S})$ with $\overline{S}$ large enough and an artificial boundary is set at $S = \overline{S}$. The boundary value problem is given by:

$$\begin{cases}
\frac{\partial u}{\partial t} - \frac{1}{2} \sigma^2(S, t) S^2 \frac{\partial^2 u}{\partial S^2} = 0 & (t, S) \in (0, T) \times (0, \overline{S}) \\
u(S, 0) = g(S) & S \in (0, \overline{S}) \\
u(\overline{S}, t) = 0 & t \in (0, T)
\end{cases}$$

The interval $[0, \overline{S}]$ is split into $N$ uniform subintervals $\kappa_i = [S_{i-1}, S_i]$, $1 \leq i \leq N + 1$. We call $h = S_i - S_{i-1}$ for all $i$ the step size and define the mesh $T_h$ of $[0, \overline{S}]$ as the set $\{\kappa_1, \ldots, \kappa_{N+1}\}$. The strike $K$ must coincide with some node of the mesh. We define the space $V_h$ by

$$V_h = \{ v_h \in C^0([0, \overline{S}], v_h(\overline{S}) = 0, \quad \forall \kappa \in T_h, v_h|_\kappa \text{ is affine} \}$$

As the strike coincides with a node of the mesh, $g \in V_h$ when $g(S) = (K - S)^+$. The time interval is split into $M$ uniform subintervals. The timestep is defined by $\Delta t$. Applying the Euler implicit scheme, the discrete problem consists in finding $(u^m_h)_{1 \leq m \leq M}$ such that $u^m_h \in V_h$ with $u^0_h(S_i) = g(S_i)$ for $1 \leq i \leq N + 1$ and for all $v_h \in V_h$:

$$\left( u^m_h - u^{m-1}_h, v_h \right) + \Delta t a_t(u^m_h, v_h) = 0$$
The nodal basis \((w_i)_{i=0,...,N}\) of \(V_h\) is composed of the piecewise linear function on the mesh \(w_i\) which takes the value 1 at \(S_i\) and 0 at \(S_j\) when \(j \neq i\). The solution can be rewritten as:

\[
u_h^m(S) = \sum_{i=0}^{N} u_h^m(S_i) w_i(S)
\]

Let \(M \in \mathbb{R}^{N+1}\) be such that \(M_{i,j} = (w_i, w_j)\) and \(A\) in \(\mathbb{R}^{N+1}\) defined by \(A_{i,j} = a_t(w_i, w_j)\) for 

\[
\begin{align*}
A_{i,i} &= \frac{S_i^2 \sigma^2(S_i,t_m)}{2h}, & M_{i,i-1} &= h/6, & 1 \leq i \leq N, \\
A_{i,i} &= \frac{S_i^2 \sigma^2(S_i,t_m)}{2h}, & M_{i,i} &= 2h/3, & 1 \leq i \leq N, \\
A_{i,i+1} &= -\frac{S_i^2 \sigma^2(S_i,t_m)}{2h}, & M_{i,i+1} &= h/6, & 0 \leq i \leq N - 1, \\
A_{0,0} &= 0, & M_{0,0} &= h/3.
\end{align*}
\]

### 3.3. The case where the diffusion coefficient only depends on time.

Suppose that \(\sigma(t,x) = \sigma(t)\), \(t \in [0,T]\), only depends on time. In that case, we may show (see [15]) that

\[
h(t,x) = \int_{-\infty}^{\infty} g \left( xe^{\rho_t y - \frac{\rho_t^2}{2}} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy,
\]

where \(\rho_t \geq 0\) is defined by \(\rho_t^2 = \int_t^T \sigma_u^2 du\). Moreover, for some particular functions \(g\), e.g. \(g(x) = (x - K)^+\), it is trivial to derive an explicit formula of \(h(t,x)\). As \(k^n-\) and \(k^n+\) do not depend on the space variable, we also deduce an explicit expression of \(h^n\), solution to (2.6):

\[
h(t_{n-i}^n, x) = \frac{1}{2i} \sum_{z \in \mathbb{Z}_{n-i}} g(xz), \quad i = 0, \cdots, n
\]

(3.10)
where the sets \((E_i)_{i=0,\ldots,n}\) do not depend on \(x\) and are recursively defined as \(E_n = \{1\}\) and \(E_{n-i-1} = (k_i^n E_{n-i-1}) \cup (k_i^n + E_{n-i})\) for all \(i = 0, \ldots, n-1\).

Regarding the implementation, we may see each \(E_i\) as a vector with \(2^{n-i}\) components. Once these vectors computed for all, we may simulate the scheme (2.6) for any \(x\) and any arbitrary terminal condition \(g\). Indeed, just compute the vectors \(G_i\) whose components are \((g(xz))_{z \in E_i}\) and deduce \(h(t_{n-i},x)\) as the scalar product of \(G_i\) with \(1 = (1, \ldots, 1) \in \mathbb{R}^{2^{n-i}}\).

Let us now consider the case \(\sigma(t) = \sigma(2 + \cos(t))\). Then, we may show that

\[
\rho_i^2 = 4\sigma^2 \left[(T + \sin T) - (t + \sin t)\right] + \frac{\sigma^2}{2} \left[(T + \frac{1}{2}\sin(2T) - (t + \frac{1}{2}\sin 2t)\right]
\]

and, for \(g(x) = (x - K)^+\), \(h(t,x)\) is explicitly given by

\[
h(t, x) = \int_{-\infty}^{\infty} g \left(xe^{\rho^2/2} - y\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy
\]

Therefore, we are in position to numerically compare the convergence error between the explicit solution given above and the approximations provided by the functional scheme and the other methods. We first consider the parameters \(\sigma = 0.05, T = 10\) and \(K = 100\). We compare the functional scheme, the finite difference method and the finite element method in Figure 1 for different values of \(x\). The time discretization is composed of 30 timesteps. The
space discretization is fixed to 30000 steps for the finite difference scheme and 40000 for the finite element scheme. Results are compared to the analytical solution. Results are good and all methods perform well. In Figure 2, we focus on the price trajectory when \( x = 120 \). Once again, results are good. All methods fit well the analytical solution.

![Graph](image)

**Fig 1.** Numerical example with \( \sigma = 0.05, T = 10, K = 100 \) and \( x \in [80; 120] \).

We now estimate the convergence error as a function of computation time. With the same parameters, we refine the time grid for the functional scheme and we refine with the same order the grids both in time and space for the other methods. Results are presented in the Figure 3. The three methods provide an error smaller than 1% in less than one second. The functional scheme provides an error of 2.51% in 0.01s while the error is 11.69% with the finite difference method in 0.01s. However, the functional scheme error convergence is slow to decrease in comparison with the other methods which provide an error of 0.1% in less than 5 seconds.

![Graph](image)

**Fig 2.** Numerical example with \( \sigma = 0.05, T = 10, K = 100, x = 120 \).
3.4. The case where the diffusion coefficient only depends on space.

Let us consider the diffusion coefficient \( \sigma(t, x) = \sigma(x) = \sigma^* (1 + e^{-x^2}) \). As \( h(t, x) \) does not admit any analytic expression, we first evaluate it using its probabilistic representation through a Monte-Carlo discretization of the associated diffusion process (5.17). The time discretization is composed of 10000 dates with 50000 samples for the Monte-Carlo method and parameters remain the same for the three methods. As we may observe in Figure 4, results are still good.

In Figure 5, \( x \) takes values in a range from 80 to 120 and results are compared to the Monte Carlo method. All methods perform well.
In Figure 6, we estimate the convergence error as a function of computation time when the parameters are $\sigma = 0.05$, $T = 10$, $K = 100$ and $x = 120$. Results are very good for the functional scheme. The convergence error is lower than 1% in less than 0.01 second. The functional scheme is more precise than the finite element and the finite difference methods for very short computation time. When computation time increases, the functional scheme and finite difference methods provide comparable error.
3.5. The case where the diffusion coefficient depends on time and space.

We consider the volatility function \( \sigma(t, x) = \sigma(1 + \frac{0.01x^2}{1+0.01t^2} + t) \). The time discretization is composed of 30 timesteps. The space discretization is composed of 30000 points for the finite difference scheme and 40000 points for the finite element scheme. Results are compared to the Monte Carlo approximation. The Monte Carlo simulation parameters are 50000 simulations and 10000 timesteps per simulation. Figure 7 shows results for parameters \( \sigma = 0.05, T = 10, K = 100 \) and \( x = 120 \). Functional scheme, finite difference method and finite element method present very similar results.
Fig 7. Numerical example with $\sigma = 0.5$, $T = 10$, $K = 100$, $x = 120$.

In Figure 8, $x$ takes values in a range from 80 to 120 and results are compared to the Monte Carlo method. All methods still perform well. The functional scheme appears to perform well whatever the diffusion function we have tested.

Fig 8. Numerical example with $\sigma = 0.05$, $T = 10$, $K = 100$ and $x \in [80; 120]$.

In Figure 9, we compare the convergence speed. The error drops below 1% in less than one second. The functional scheme performs slightly better than the finite difference method. The finite element method appears to be very efficient.

Notice that it is possible to execute parallel computations by means of programming interfaces (e.g. OpenMP or MPI) i.e. we may use several calculus units simultaneously and reduce computation time. Using $n \geq 1$ processors, we should theoretically divide the needed time by $n$. In practice, that depends on how the executed parallel calculus are interconnected. In the finite
difference method, when computing the value function at some point of the grid, we need to estimate some expressions depending on the adjacent nodes that may be calculated and used by other processors. This should increase the computation time, as it is necessary to wait for some computations to be executed before starting new ones. On the contrary, for the functional scheme, we do not face this problem as the nodes of the binomial tree at the same level are independent. This is confirmed by the realized time savings we observe to be very closed to the theoretical ones (e.g. 50% with two cores and 75% with 4 cores), see Table 10 with the current example and the parameters $\sigma = 0.5$, $T = 10$, $K = 100$, $x = 120$. Therefore, with parallel computing, the functional scheme should significantly outperform the other methods in the examples we have presented.

<table>
<thead>
<tr>
<th>Number of cores</th>
<th>Computation time (s)</th>
<th>Theoretical time saving</th>
<th>Realized time saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>388.203</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>194.492</td>
<td>50%</td>
<td>49.90%</td>
</tr>
<tr>
<td>4</td>
<td>102.717</td>
<td>75%</td>
<td>73.54%</td>
</tr>
</tbody>
</table>

**Fig 9.** Numerical example with $\sigma = 0.05$, $T = 10$, $K = 100$, $x = 120$. Error as a function of computation time.

**Fig 10.** Time savings with the parallel functional scheme when $\sigma = 0.5$, $T = 10$, $K = 100$, $x = 120$. 
3.6. The case of an unbounded payoff function.

Let us consider the payoff function \( h(x) = (x - K)^8 \) and let us fix the following parameters: \( \sigma = 0.25 \), \( T = 1 \) and \( K = 0.5 \). We then compute \( t \mapsto h(t, x) \) for \( x = 2 \). We compare the obtained results to the approach based on the probabilistic representation of \( h(t, x) \), i.e. by implementing the Monte Carlo (MC) technique with a time discretization of 50000 dates and a sample of 100000 trajectories.

In Figure 11, the (FD) method performs as the error is 1% in less than one second. However, the method doesn’t converge and the error remains constant even if we increase the discretization. The (FS) method provides an error of 1% in one second, and for larger computation time, the error is smaller than the (FD) method one. In Figure 12, we observe how the error decreases in function of the number of timesteps. We get an error of 0.09% error in seven minutes.

![Fig 11. Numerical example with \( \sigma = 0.05 \), \( T = 10 \), \( K = 100 \), \( x = 120 \). Error estimation as a function of computation time.](image)

<table>
<thead>
<tr>
<th>number of iterations</th>
<th>Functional Scheme</th>
<th>Time (s)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>341.277</td>
<td>0.047</td>
<td>11.54%</td>
</tr>
<tr>
<td>20</td>
<td>345.047</td>
<td>0.195</td>
<td>10.56%</td>
</tr>
<tr>
<td>22</td>
<td>387.333</td>
<td>1.604</td>
<td>0.40%</td>
</tr>
<tr>
<td>24</td>
<td>386.950</td>
<td>6.116</td>
<td>0.30%</td>
</tr>
<tr>
<td>26</td>
<td>386.630</td>
<td>25.831</td>
<td>0.22%</td>
</tr>
<tr>
<td>28</td>
<td>386.358</td>
<td>101.185</td>
<td>0.12%</td>
</tr>
<tr>
<td>30</td>
<td>386.125</td>
<td>428.183</td>
<td>0.09%</td>
</tr>
</tbody>
</table>

![Fig 12. Approximation of \( h(0, x) \) with the functional scheme.](image)
We now consider the volatility function

\[ \sigma(t, x) = \sigma \left( 1 + \frac{x^2}{100 \left( 1 + \frac{x^2}{100} \right)} + t \right). \]

We observe that results are very sensitive to the space variable and its discretization. Reducing the error requires to significantly refine the time grid.

Fig 13. Numerical example with \( \sigma = 0.05, T = 10, K = 100, x = 120 \). Error estimation.

Clearly, on this example the (FS) method seems to outperform significantly the two other ones, see Figure 13. Moreover, contrarily to (FD) and (FE) methods, it is possible to reduce the error as much as desired, see Table 14.

<table>
<thead>
<tr>
<th>number of iterations</th>
<th>Functional Scheme</th>
<th>Time (s)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>6135.580</td>
<td>0.014</td>
<td>54.72%</td>
</tr>
<tr>
<td>18</td>
<td>6803.324</td>
<td>0.065</td>
<td>49.85%</td>
</tr>
<tr>
<td>20</td>
<td>7425.858</td>
<td>0.269</td>
<td>45.26%</td>
</tr>
<tr>
<td>22</td>
<td>12222.099</td>
<td>2.169</td>
<td>9.96%</td>
</tr>
<tr>
<td>24</td>
<td>12684.706</td>
<td>8.588</td>
<td>6.49%</td>
</tr>
<tr>
<td>26</td>
<td>13110.191</td>
<td>34.967</td>
<td>3.36%</td>
</tr>
<tr>
<td>28</td>
<td>13688.050</td>
<td>117.510</td>
<td>0.46%</td>
</tr>
</tbody>
</table>

Fig 14. Approximation of \( h(0, x) \) with the functional scheme.

**Conclusion:** Through the numerical experiments we propose, we observe that the performance of the three methods of consideration are similar for diffusion functions only depending on time or space variable and with reasonable payoff functions. Although, the functional scheme seems to outperform the two other methods as soon as we consider more sophisticated diffusion
functions, e.g. depending both on time and space variable. Moreover, the outperformance may be significative for non linear growing payoff functions. At last, as parallel computing is very well adapted to the functional scheme, it is possible to improve even more its efficiency.

4. Proof

From a line to the next one, we shall use the same notation $C$ for some distinct constants independent of $n$. Also, notice that there is a constant $M > 0$ such that $|k_i^{n^+}| + |k_i^{n^-}| \leq M$ for all $n$ and $t \in [0,T]$. We also use the notation $\Delta t^n_i = t^n_i - t^{n-1}_i = T/n, i \leq n$. At last, observe that the results below are trivial when $x = 0$ hence we only provide the proofs when $x \neq 0$.

Lemma 4.1. Suppose that the conditions of Theorem (2.1) hold. Then, there exists a constant $C > 0$ which does not depend on $x$ such that

$$\sup_{t \in [t^n_{n-1}, T]} |h^n(t, x) - h(t, x)| \leq \frac{C|x|}{\sqrt{n}}.$$  

Proof. Let us define $\delta^n_t = |h(t, x) - h^n(t, x)|$. Recall that $h(t, x) = \mathbb{E}g(S_{x,t}(T))$ where $S_{x,t}$ is the solution to (5.17). We have:

$$\delta^n_t = |h(t, x) - \frac{1}{2}g(k_{n-1}^{n^-} x) - \frac{1}{2}g(k_{n-1}^{n^+} x)|$$

$$\leq |h(t, x) - g(x)| + \frac{1}{2} |g(k_{n-1}^{n^-} x) - g(x)| + \frac{1}{2} |g(k_{n-1}^{n^+} x) - g(x)|$$

$$\leq \sup_{t \in [t^n_{n-1}, T]} |h(t, x) - g(x)| + \frac{L_g}{2} (1 - k_{n-1}^{d}) |x| + \frac{L_g}{2} (k_{n-1}^{u} - 1) |x|$$

$$\leq L_g \sup_{t \in [t^n_{n-1}, T]} \mathbb{E} |S_{x,t}(T) - x| + \frac{L_g}{2} \sigma(t, x) \sqrt{\frac{T}{n}} |x| + \frac{L_g}{2} \sigma(t, x) \sqrt{\frac{T}{n}} |x|$$

$$\leq C|x| \sqrt{\frac{T}{n}} + \frac{L_g}{2} \sigma^* \sqrt{\frac{T}{n}} |x|,$$

where the last inequality is deduced from Lemma 5.1 and $\sigma^* = \sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t, x)|$.

\[\square\]

Lemma 4.2. Suppose that the conditions of Theorem (2.1) hold. Then, there exists a constant $C > 0$ such that

$$|h^n(t^n_{n-2}, x) - h(t^n_{n-2}, x)| \leq \frac{C|x|}{\sqrt{n}}.$$
Proof. Let us introduce $\delta^n_{t-2} = |h^n(t_{n-2}, x) - h(t_{n-2}, x)|$. We have

$$\delta^n_{t-2} = \left| \frac{1}{2} h^n(t_{n-1}, k_{n-2}^- x) + \frac{1}{2} h^n(t_{n-1}, k_{n-2}^+ x) - h(t_{n-2}, x) \right|,$$

$$\leq \frac{1}{2} \left| h^n(t_{n-1}, k_{n-2}^- x) - h(t_{n-2}, x) \right| + \frac{1}{2} \left| h^n(t_{n-1}, k_{n-2}^+ x) - h(t_{n-2}, x) \right|$$

$$\leq \frac{1}{2} \left| h^n(t_{n-1}, k_{n-2}^- x) - h(t_{n-1}, k_{n-2}^- x) \right| + \frac{1}{2} \left| h^n(t_{n-1}, k_{n-2}^+ x) - h(t_{n-1}, k_{n-2}^+ x) \right|$$

$$+ \frac{1}{2} \left| h(t_{n-2}, k_{n-2}^- x) - h(t_{n-2}, k_{n-2}^- x) \right| + \frac{1}{2} \left| h(t_{n-2}, k_{n-2}^+ x) - h(t_{n-2}, k_{n-2}^+ x) \right|$$

Under Condition D, by the mean value theorem, we deduce that

$$| h(t_{n-2}^n, x) - h(t_{n-2}^n, k_{n-2}^- x) | \leq C \left| x - k_{n-2}^- x \right| \leq C \left| x \right| \cdot 1 - k_{n-2}^- 2,$$

$$\leq C \left| x \right| \sqrt{T / n},$$

$$| h(t_{n-2}^n, x) - h(t_{n-2}^n, k_{n-2}^+ x) | \leq C \left| x - k_{n-2}^+ x \right| \leq C \left| x \right| \cdot 1 - k_{n-2}^+ 2,$$

$$\leq C \left| x \right| \sqrt{T / n}.$$

Note that, under Condition D, we have

$$| h(t, x) | = \frac{x^2 \sigma^2(t, x)}{2} \left| h_{xx}(t, x) \right| \leq \frac{C \left| x \right|}{\sqrt{T - t}}. \quad (4.11)$$

Therefore, by the mean value theorem, since $k_{n-2}^n, k_{n-2}^+ \leq M$,

$$| h(t_{n-1}^n, k_{n-2}^- x) - h(t_{n-2}^n, k_{n-2}^- x) | \leq \frac{CM \left| x \right|}{\sqrt{T - t_{n-1}^n}} \Delta t_{n-1}^n \leq CM \left| x \right| \sqrt{T / n},$$

$$| h(t_{n-1}^n, k_{n-2}^+ x) - h(t_{n-2}^n, k_{n-2}^+ x) | \leq \frac{CM \left| x \right|}{\sqrt{T - t_{n-1}^n}} \Delta t_{n-1}^n \leq CM \left| x \right| \sqrt{T / n}.$$

At last, by Lemma 4.1,
\[ \gamma_n(t^n_{n-1}, x) := |h^n(t^n_{n-1}, k^n_{n-2} x) - h(t^n_{n-1}, k^n_{n-2} x)| \leq C k^n_{n-2} |x| \sqrt{T/n}, \]

\[ \gamma^n(t^n_{n-1}, x) := |h^n(t^n_{n-1}, k^n_{n-2} x) - h(t^n_{n-1}, k^n_{n-2} x)| \leq C k^n_{n-2} |x| \sqrt{T/n}. \]

As \( k^n_{n-2} + k^n_{n-2} = 2 \), we deduce that

\[ \frac{1}{2} \gamma^n(t^n_{n-1}, x) + \frac{1}{2} \gamma^n(t^n_{n-1}, x) \leq C |x| \sqrt{T/n}. \]

The conclusion follows. \( \square \)

**Lemma 4.3.** Suppose that the conditions of Theorem (2.1) hold. Then, there exists a constant \( C > 0 \) such that

\[ |h^n(t^n_i, x) - h(t^n_i, x)| \leq \frac{C|x|}{\sqrt{n}}, \quad \text{for all } i \leq n - 2. \]

**Proof.** We first prove the result for \( i = n - 2 \) and, then, we generalize the result by induction. Let us introduce the function

\[ F_i(x) = \frac{1}{2} h(t^n_i, k^n_{i-1} x) + \frac{1}{2} h(t^n_i, k^n_{i-1} x). \]

By the Taylor formula, we get that

\[ F_i(x) = \frac{1}{2} \left[ h(t^n_{i-1}, x) + R^{11}_i(x) + R^{12}_i(x) + R^{13}_i(x) \right] \]

\[ + \frac{1}{2} \left[ h(t^n_{i-1}, x) + R^{21}_i(x) + R^{22}_i(x) + R^{23}_i(x) \right]. \]

where the residuals terms \( (R^{jm}_i)_{j=1,2,m=1,2,3} \) are defined as follows. There exists some constants \( \alpha_i, i = 1, \cdots, 4 \), and some variables \( \tilde{t}^n_{i-1} \in [t^n_{i-1}, t^n_i], \tilde{x} = x + \tilde{x}(k^n_{i-1} - 1), i = 1, \cdots, 4 \), and some variables \( \tilde{t}^n_{i-1} \in [t^n_{i-1}, t^n_i], \tilde{x} = x + \tilde{x}(k^n_{i-1} - 1), \)

\[ \tilde{t}^n_{i-1} \in [t^n_{i-1}, t^n_i], \tilde{x} = x + \tilde{x}(k^n_{i-1} - 1), \]
\[ \hat{\theta} \in [0, 1], \] such that

\[
R_{i}^{11}(x) = h_t(t^n_{i-1}, x) \frac{T}{n} - h_x(t^n_{i-1}, x)x\sigma(t^n_{i-1}, x)\sqrt{\frac{T}{n}},
\]

\[
R_{i}^{12}(x) = -h_{tx}(t^n_{i-1}, x)x\sigma(t^n_{i-1}, x) \left( \frac{T}{n} \right)^\frac{3}{2} + \frac{1}{2} h_{xx}(t^n_{i-1}, x)x^2\sigma^2(t^n_{i-1}, x) \frac{T}{n}
\]

\[ + \frac{1}{2} h_{tt}(t^n_{i-1}, x) \left( \frac{T}{n} \right)^2, \]

\[
R_{i}^{13}(x) = \alpha_1 h_{ttt}(\hat{t}^n_{i-1}, \hat{x}) \left( \frac{T}{n} \right)^3 - \alpha_2 h_{ttx}(\hat{t}^n_{i-1}, \hat{x})\sigma(t^n_{i-1}, x) \left( \frac{T}{n} \right)^{5/2}
\]

\[ + \alpha_3 h_{txx}(\hat{t}^n_{i-1}, \hat{x})\sigma^2(t^n_{i-1}, x)x^2 \left( \frac{T}{n} \right)^2 - \alpha_4 h_{xxx}(\hat{t}^n_{i-1}, \hat{x})\sigma^3(t^n_{i-1}, x)x^3 \left( \frac{T}{n} \right)^{3/2}, \]

\[
R_{i}^{21}(x) = h_t(t^n_{i-1}, x) \frac{T}{n} + h_x(t^n_{i-1}, x)x\sigma(t^n_{i-1}, x)\sqrt{\frac{T}{n}},
\]

\[
R_{i}^{22}(x) = h_{tx}(t^n_{i-1}, x)x\sigma(t^n_{i-1}, x) \left( \frac{T}{n} \right)^\frac{3}{2} + \frac{1}{2} h_{xx}(t^n_{i-1}, x)x^2\sigma^2(t^n_{i-1}, x) \frac{T}{n}
\]

\[ + \frac{1}{2} h_{tt}(t^n_{i-1}, x) \left( \frac{T}{n} \right)^2, \]

\[
R_{i}^{13}(x) = \alpha_1 h_{ttt}(\hat{t}^n_{i-1}, \hat{x}) \left( \frac{T}{n} \right)^3 + \alpha_2 h_{ttx}(\hat{t}^n_{i-1}, \hat{x})\sigma(t^n_{i-1}, x) \left( \frac{T}{n} \right)^{5/2}
\]

\[ + \alpha_3 h_{txx}(\hat{t}^n_{i-1}, \hat{x})\sigma^2(t^n_{i-1}, x)x^2 \left( \frac{T}{n} \right)^2 + \alpha_4 h_{xxx}(\hat{t}^n_{i-1}, \hat{x})\sigma^3(t^n_{i-1}, x)x^3 \left( \frac{T}{n} \right)^{3/2}. \]

We deduce that \( F_i(x) = h(t^n_{i-1}, x) + R_{i}^{11}(x) + R_{i}^{22}(x) + R_{i}^{13}(x) \) where \( R_{i}^{1}(x) = \frac{1}{2}(R_{i}^{11}(x) + R_{i}^{21}(x)), R_{i}^{2}(x) = \frac{1}{2}(R_{i}^{12}(x) + R_{i}^{22}(x)) \) and \( R_{i}^{3}(x) = \frac{1}{2}(R_{i}^{13}(x) + R_{i}^{23}(x)). \)

As \( h \) is the solution to (2.4), we may simplify \( F_i(x) \) as

\[
F_i(x) = h(t^n_{i-1}, x) + \frac{1}{2} h_{tt}(t^n_{i-1}, x) \left( \frac{T}{n} \right)^2 + R_{i}^{3}(x).
\]

Under Condition D, we may show the existence of a constant \( C \) independent of \( i \) and \( n \) such that

\[
\frac{1}{2} |h_{tt}(t^n_{i-1}, x)| \leq \frac{C |x|}{(T - t^n_{i-1})^{3/2}}, \tag{4.12}
\]
On the other hand, since \( \tilde{t}_{i-1}^n, \hat{t}_{i-1}^n \in [t_{i-1}^n, t_i^n] \) and \(|\hat{x}| + |\bar{x}| \leq C|x|\) where \( C > 0 \) is independent of \( i \) and \( n \), the residual error \( R_i^2(x) \) is the sum of terms which may be dominated under Condition D using the following inequalities:

\[
\begin{align*}
|\alpha_1 h_{ttt}(\hat{t}_{i-1}^n, \hat{x})| & \leq \frac{C |x|}{(T - t_i^n)^{5/2}}. \quad (4.13) \\
|\alpha_2 h_{xtt}(\hat{t}_{i-1}^n, \hat{x})\sigma(t_{i-1}^n, x)x| & \leq \frac{C |x|}{(T - t_i^n)^2}. \quad (4.14) \\
|\alpha_3 h_{xxx}(\hat{t}_{i-1}^n, \hat{x})\sigma^2(t_{i-1}^n, x)x^2| & \leq \frac{C |x|}{(T - t_i^n)^{3/2}}. \quad (4.15)
\end{align*}
\]

The inequalities above are also satisfied if we replace \( \tilde{t}_{i-1}^n \) by \( \hat{t}_{i-1}^n \) and \( \hat{x} \) by \( \bar{x} \).

Recall that the constant \( C \) may denote distinct constants that change for a line to the next one but these constants do not depend on \( i \) and \( n \). At last, by the Taylor formula, we rewrite the last term as

\[
\epsilon_i = \alpha_4 \left( h_{xxx}(\hat{t}_{i-1}^n, \hat{x})\sigma^3(t_{i-1}^n, x) - h_{xxx}(\tilde{t}_{i-1}^n, \bar{x})\sigma^3(t_{i-1}^n, x) \right) x^3 \left( \frac{T}{n} \right)^{3/2},
\]

\[
= \alpha_4 h_{xxxt}(\hat{t}_{i-1}^n, \hat{x})x^3(\hat{t}_{i-1}^n - \tilde{t}_{i-1}^n) \left( \frac{T}{n} \right)^{3/2} + \alpha_4 h_{xxx}(\tilde{t}_{i-1}^n, \bar{x})x^3(\hat{x} - \bar{x}) \left( \frac{T}{n} \right)^{3/2},
\]

where \( \tilde{t}_{i-1}^n \in [\hat{t}_{i-1}^n, \hat{t}_{i-1}^n] \) and \( \bar{x} = \alpha \hat{x} + (1 - \alpha)\hat{x} \) for some \( \alpha \in [0, 1] \). It is easily seen that \(|\tilde{t}_{i-1}^n - \hat{t}_{i-1}^n| \leq CT/n \) for some constant \( C \) independent of \( i \) and \( n \). Similarly, by definition of \( \hat{x} \) and \( \bar{x} \), there exists a constant \( C \) such that \(|\hat{x} - \bar{x}| \leq C|x|\sqrt{T/n}\). Therefore,

\[
|\epsilon_i| \leq \frac{C |x|}{(T - t_i^n)^2} \left( \frac{T}{n} \right)^{5/2} + \frac{C |x|}{(T - t_i^n)^{3/2}} \left( \frac{T}{n} \right)^2. \quad (4.16)
\]

Let us introduce \( \delta_{i-1}^n(x) := |h^n(t_{i-1}^n, x) - h(t_{i-1}^n, x)|, \ i \leq n - 2. \) From
above, we get that
\[
\delta_{t_{i-1}}^n(x) = \left| \frac{1}{2} h^n(t_i^n, k_{i-1}^{n-}x) + \frac{1}{2} h^n(t_i^n, k_{i-1}^{n+}x) - h(t_{i-1}^n, x) \right|
\]
\[
= \left| \frac{1}{2} h^n(t_i^n, k_{i-1}^{n-}x) + \frac{1}{2} h^n(t_i^n, k_{i-1}^{n+}x) - F_i(x) + \frac{1}{2} h(t_{i-1}^n, x) \left( \frac{T}{n} \right)^2 + R_i^3(x) \right|
\]
\[
\leq \frac{1}{2} \delta_{i}^n(k_{i-1}^{n-}x) + \frac{1}{2} \delta_{i}^n(k_{i-1}^{n+}x) + \frac{1}{2} \left( \frac{T}{n} \right)^2 \left| h(t_{i-1}^n, x) \right| + \left| R_i^3(x) \right|.
\]

Using the inequalities (4.12), (4.13),\cdots, (4.16), we then deduce a constant $C > 0$ independent of $i$ and $n$ such that
\[
\delta_{i}^n(x) \leq \frac{1}{2} \delta_{i}^n(k_{i-1}^{n-}x) + \frac{1}{2} \delta_{i}^n(k_{i-1}^{n+}x)
\]
\[
+ \frac{C|x|}{(T - t_i^n)^{\frac{3}{2}}} \left( \frac{T}{n} \right)^{\frac{3}{2}} + \frac{C|x|}{(T - t_i^n)^{\frac{5}{2}}} \left( \frac{T}{n} \right)^{\frac{5}{2}}
\]
\[
+ \frac{C|x|}{(T - t_i^n)^2} \left( \frac{T}{n} \right)^2 + \frac{C|x|}{(T - t_i^n)^{\frac{5}{2}}} \left( \frac{T}{n} \right)^{\frac{5}{2}}.
\]

Recall that, by Lemma 4.2,
\[
\frac{1}{2} \left( \delta_{i-2}^n(k_{i-2}^{n-}x) + \delta_{i-2}^n(k_{i-2}^{n+}x) \right) \leq \frac{1}{2} C|x|(k_{n-3}^{n-} + k_{n-3}^{n+}) \sqrt{\frac{T}{n}} = C|x| \sqrt{\frac{T}{n}}.
\]
We deduce that
\[
|\delta_{i-3}^n(x)| \leq C|x| \sqrt{\frac{T}{n}} + \frac{C|x|}{(T - t_{i-2}^n)^{\frac{3}{2}}} \left( \frac{T}{n} \right)^{\frac{3}{2}} + \frac{C|x|}{(T - t_{i-2}^n)^{\frac{5}{2}}} \left( \frac{T}{n} \right)^{\frac{5}{2}}
\]
\[
+ \frac{C|x|}{(T - t_{i-2}^n)^2} \left( \frac{T}{n} \right) + \frac{C|x|}{(T - t_{i-2}^n)^{\frac{5}{2}}} \left( \frac{T}{n} \right)^{\frac{5}{2}}.
\]

Repeating the reasoning, given that $\frac{1}{2}(k_{i-1}^{n-} + k_{i-1}^{n+}) = 1$, we deduce by induction that, for every $i \leq n - 3$,
\[
|\delta_{i}^n(x)| \leq C|x| \sqrt{\frac{T}{n}} + S_{i,n}^1 + S_{i,n}^2 + S_{i,n}^3 + S_{i,n}^4.
\]
where, for \( i \leq n - 3 \),

\[
S_{i,n}^1 := \sum_{j=i+1}^{n-2} \frac{C|x|}{(T-t_j^n)^{3/2}} \left( \frac{T}{n} \right)^{3/2}, \quad S_{i,n}^2 := \sum_{j=i+1}^{n-2} \frac{C|x|}{(T-t_j^n)^{3/2}} \left( \frac{T}{n} \right)^{3/2},
\]

\[
S_{i,n}^3 := \sum_{j=i+1}^{n-2} \frac{C|x|}{(T-t_j^n)^{2}} \left( \frac{T}{n} \right)^{2}, \quad S_{i,n}^4 := \sum_{j=i+1}^{n-2} \frac{C|x|}{(T-t_j^n)^{2}} \left( \frac{T}{n} \right)^{2}.
\]

Since \((T-u)^{-1} \geq (T-t_j^n)^{-1}\) if \( u \in [t_j^n, t_{j+1}^n]\), we deduce that

\[
S_{i,n}^1 \leq C|x| \sqrt{T/n} \int_0^T \frac{1}{\sqrt{T-t}} \, dt \leq C|x| \sqrt{T/n},
\]

\[
S_{i,n}^2 \leq C|x| \frac{T}{n} \int_0^{T/T_n} \frac{1}{(T-t)^{3/2}} \, dt \leq C|x| \sqrt{T/n},
\]

\[
S_{i,n}^3 \leq C|x| \left( \frac{T}{n} \right)^{3/2} \int_0^{T/T_n} \frac{1}{(T-t)^2} \, dt \leq C|x| \sqrt{T/n},
\]

\[
S_{i,n}^4 \leq C|x| \left( \frac{T}{n} \right)^{2} \int_0^{T/T_n} \frac{1}{(T-t)^{5/2}} \, dt \leq C|x| \sqrt{T/n}.
\]

Both with Lemma 4.1 and 4.2 and the inequalities above, we may conclude.

\[ \square \]

**Corollary 4.4.** Suppose that the conditions of Theorem (2.1) hold. Then, there exists a constant \( C > 0 \) such that

\[
\sup_{t \in [0,T]} |h^n(t, x) - h(t, x)| \leq \frac{C|x|}{\sqrt{n}}.
\]

**Proof.** Let \( t \in [0,T] \) be such that \( t \in [t_{i-1}^n, t_i^n) \) for some \( i \geq 1 \). By Lemma 4.1, we may suppose that \( i \leq n - 2 \). Then,

\[
|h^n(t, x) - h(t, x)| = |h^n(t_{i-1}^n, x) - h(t, x)| \leq |h^n(t_{i-1}^n, x) - h(t_{i-1}^n, x)| + |h(t_{i-1}^n, x) - h(t, x)|.
\]

By the mean value theorem and Inequality (4.11), since \( T - t_i^n \geq T/n \), we deduce that

\[
|h(t_{i-1}^n, x) - h(t, x)| \leq \frac{C|x|}{\sqrt{T-t_i^n}} \Delta t_i^n \leq \frac{C|x|}{\sqrt{n}}.
\]

We then conclude by Lemma 4.3. \( \square \)
5. Appendix

Let us consider the unique solution $S_{x,t}$, $t \in [0,T]$, to the stochastic differential equation

$$dS_{t,x}(u) = S_{t,x}(u)\sigma(u, S_{t,x}(u))dW_u, \quad u \in [t,T], \quad S_{t,x}(t) = x \in \mathbb{R}, \quad (5.17)$$

where $W$ is a standard Brownian motion.

**Lemma 5.1.** Suppose that $t \in [0,T]$. Let $S_{x,t}$ be the solution to the stochastic differential equation $(5.17)$. If $\sigma$ is bounded by a constant $\sigma^* > 0$, there exists a constant $C$ such that

$$\mathbb{E} \sup_{t \leq u \leq T} S_{t,x}^2(u) \leq Cx^2, \quad \mathbb{E}(S_{t,x}(T) - x)^2 \leq Cx^2(T - t).$$

**Proof.** By the Doob’s inequality, we obtain that for every $t \leq r \leq T$:

$$\phi(r) := \mathbb{E} \left| \sup_{t \leq u \leq r} S_{x,t}(u) \right|^2 \leq 4\mathbb{E} \left| S_{x,t}(r) \right|^2$$

As $S_{x,t}(r) = x + \int_t^r \sigma(u, S_{x,t}(u))S_{x,t}(u)dW_u$, using the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and the Ito isometry, we get that:

$$\phi(r) \leq 8x^2 + 8\mathbb{E} \left( \int_t^r \sigma(u, S_{x,t}(u))S_{x,t}(u)dW_u \right)^2$$

$$\leq 8x^2 + 8\mathbb{E} \left( \int_t^r \sigma^2(u, S_{x,t}(u))S_{x,t}^2(du) \right)$$

$$\leq 8x^2 + 8(\sigma^*)^2 \int_t^r \mathbb{E} S_{x,t}^2(du)$$

$$\leq 8x^2 + 8(\sigma^*)^2 \int_t^r \phi(u)du.$$

Applying the Gronwall lemma, we deduce that:

$$\mathbb{E} \left| S_{x,t}(r) \right|^2 \leq 8x^2 \exp(8(\sigma^*(T - t)) \leq Cx^2,$$

where $C$ does not depend on $x$. By the Ito isometry, we then deduce that

$$\mathbb{E}(S_{x,t}(T) - x)^2 = \mathbb{E} \int_t^T \sigma^2(u, S_{x,t}(u))S_{x,t}^2(du)$$

Using the inequality above, we may conclude. □
References


